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THE ARCHIMEDES STANDARDS

Model PreK-12 State Mathematics Standards

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NATIONAL

ASSOCIATION

of SCHOLARS

Freedom™
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Mission

The National Association of Scholars is an independent membership association of academics and others working to sustain the tradition of reasoned scholarship and civil debate in America's colleges and universities. We uphold the standards of a liberal arts education that fosters intellectual freedom, searches for the truth, and promotes virtuous citizenship.

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Mission

Promoting freedom in education by equipping parents & teachers. We endeavor to find, create, and promote solutions to the most pressing and destructive issues facing education in America today. Freedom in Education is committed to restoring parental rights, high-quality education, and civic virtue to our public schools by enhancing and improving content transparency, curriculum quality, and learning options and equipping parents to act.

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Learn more by visiting https://freedomined.org/.

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The Alexander Hamilton Institute for the Study of Western Civilization
California Policy Center
Californians for Equal Rights Foundation
Child Protection League Education
Civics Alliance
Connecticut Association of Scholars
Delaware Association of Scholars
Encounter Books
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Introduction

The Spirit of Archimedes

Mathematics is and always has been woven into the very fabric of the universe. It structures our world and provides a stable foundation for other fields of knowledge. It can be seen all around us—Fibonacci series in pinecones, logarithmic spirals in snails, fractals in human lungs, and π in the patterned stripes of zebras—and it helps us to wonder at the beauty of the natural world.

At the same time, math is also a human endeavor that equips individuals and cultures with powerful tools to understand and improve our world. We use math in our careers—employing financial data to make investments, biomathematics to predict the behavior of living organisms, and optimization algorithms to maximize efficiency and performance while minimizing costs and resource use. In our leisure, we use math to compare our favorite players' batting averages, to play games of strategy, and to design technology that can bring people around the world together.

Few have understood and embodied these two complementary sides of mathematics better than Archimedes, the great Greek mathematician of antiquity. Archimedes dedicated his life to math, deriving an early approximation of π, discovering the convex polyhedra that bear his name, and prefiguring modern calculus by using infinitesimals and the method of exhaustion to find the area of curved shapes. Plutarch writes of Archimedes' devotion to the field of mathematics, describing him as a man who "placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life."[1]

Yet Archimedes was also a man of action, using his mathematical knowledge to develop a series of scientific inventions to improve the quality of life in his native city of Syracuse and to defend it from invasion. Archimedes was not content to restrict his pursuit of math to the theoretical realm; rather, he employed it for engineering purposes (e.g., Archimedes' screw), for military innovations (e.g., Archimedes' claw), and even for leisure (e.g., *The Sand Reckoner*). He was moved both by the transcendent beauty and practical power of mathematics.

The teaching and learning of math ought to be done in the spirit of Archimedes. We should encourage students to delve deeply into the core principles of math, establishing a sure foundation in the concepts, processes, formulas, and practices discovered by great mathematicians throughout

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[1] Plutarch, B. Y. *Parallel lives*. Start Publishing LLC, 2012, https://www.italophiles.com/plutarch_lives.pdf.

history. And we should encourage students to apply their knowledge, thinking analytically about the world around them and solving practical problems for the public good in their professional careers and in their daily lives. It is this spirit that animates *The Archimedes Standards*. Such a spirit, embodied by Archimedes himself, values and pursues mathematics both for the sake of its timeless truth and for the good its application can bring about for our neighbors, our countrymen, and all humanity.

A Vision for Universal Excellence

State education standards are the most important guideposts for K-12 instruction in America's schools. They describe what should be taught at each grade level, providing direction for the content learned in the classroom while leaving teachers significant leeway in the methods of instruction. State standards ought to serve a variety of educational institutions, providing public schools (both charter and non-charter), private schools, and homeschools with a framework to promote quality mathematics education for all students.

Education standards should advance two goals: intellectual achievement and practical accessibility. On the one hand, standards must be made up of the best and most lasting ideas and methods, setting a high bar that challenges teachers and students to excellence. This commitment to achievement is the only way to ensure that students will be thoroughly prepared for college, career, and civic participation. On the other hand, standards must be achievable for everyone, and thus must set a basic level of competency that is in the reach of every student whatever their background, culture, or level of preparation. A solid foundation in math will serve all American students, regardless of whether they graduate from high school or go on to college. A proper set of standards, therefore, should lay out a clear and realistic path to excellence at every grade level that can be implemented across the country, in a wide variety of classrooms and contexts.

Standards can err by pursuing one of these goals at the expense of the other. They can be so ambitious that they become inaccessible to the majority of students, offering direction for an exclusive group of well-resourced schools while leaving their less advantaged counterparts struggling to manage a set of unwieldy demands. But they can also become overly relaxed, emphasizing equity to the point that they compromise the very achievement they are supposed to encourage. Indeed, lax education standards can fail to provide adequate preparation for students, making it difficult for them to succeed at the next stage of their lives. A proper set of education standards looks to avoid both of these pitfalls, mapping an attainable pathway to excellence for all students that equips the next generation to achieve its full potential.

The Need for New State Mathematics Standards

America's existing mathematics standards unfortunately promote neither high achievement nor democratic accessibility.[2] While a few states, such as Florida, have greatly improved their standards, 41 states still model their math standards on the counterproductively mediocre Common Core State

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2   A generation ago, states such as California also possessed fine mathematics standards. *Mathematics Content Standards for California Public Schools: Kindergarten Through Grade 12* (California Department of Education, 1999), https://www.cde.ca.gov/be/st/ss/docu-ments/mathstandards.pdf. Unfortunately, most states with good mathematics standards discarded them for Common Core-inspired standards.

Standards for Mathematics (CCSSM).[3] While the CCSSM contains some useful elements, on the whole, it provides a vague outline of content knowledge, it lacks rigor, and it was rushed into public use without sufficient testing and evaluation.[4] One K-12 teacher puts it this way:

> To call what [the Common Core] focuses on 'understanding' is both misleading and wrong, and there's a clear trend showing persistent loss of procedural proficiency among our students as a result. The end result of the Common Core-aligned math curriculum is STEM-deficiency rather than STEM-proficiency.[5]

The most cogent critique of the CCSSM, however, is that American mathematics education has not benefited from its adoption. American students' test scores on national and international tests have declined in recent years. On the National Assessment of Educational Progress (NAEP), American test scores have shown sustained declines since at least 2019; the lowest-performing 25% of students have shown marked declines since about 2015.[6] American students also have seen their average scores plummet on the Programme for International Student Assessment (PISA) mathematics test: average American scores dropped from 478 in 2018 to 465 in 2022, below the average for developed countries.[7] American per capita spending on education, meanwhile, remains substantially above the developed country average.[8]

This decay of mathematical competency in America affects higher education as well. In 2019, 39% of high school graduates in America were prepared for undergraduate mathematics courses; in 2022, that proportion had declined to 31%.[9] With such a scarcity of prepared American students, many universities have turned elsewhere to fill their math departments: in 2021, 54% of American doctoral degrees in mathematics and statistics were awarded to temporary visa holders.[10] While these foreign-born students should be commended for their industry, it is clear that American K-12 math education does not prepare American students to flourish in undergraduate or graduate math education.

Finally, the crisis in America's mathematics education system has sparked a mass exodus of teachers, especially elementary teachers, from the teaching profession. More teachers are leaving the classroom than ever before, and the attrition rates for new teachers leaving their schools are especially

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3   B.E.S.T. Standards for Mathematics, Florida Department of Education, https://www.fldoe.org/academics/standards/subject-areas/math-science/mathematics/; Ze'ev Wurman, David M. Steiner, Ashley Rogers, Berner, and R. James Milgram, "Better than Common Core: Florida's New K-12 Standards Raise the Bar," *Independent Institute*, July 2, 2020, https://www.independent.org/publications/article.asp?id=13176; Mathematics Standards, Common Core Standards Initiative, https://www.thecorestandards.org/Math/; Common Core States 2024, World Population Review, https://worldpopulationreview.com/state-rankings/common-core-states.

4   Peter Wood, ed., *Drilling Through the Core: Why Common Core is Bad for America*. Pioneer Institute, 2015.

5   Anonymous, "Guest Article: Why I Oppose the Common Core State Standards in Mathematics," *EduIssues*, March 7, 2020, https://eduissues.com/2020/03/07/guest-article-why-i-oppose-the-common-core-state-standards-in-mathematics/.

6   Jill Barshay, "A dismal report card in math and reading," *The Hechinger Report*, Janury 29, 2025, https://hechingerreport.org/naep-test-2024-dismal-report/; 2024 Mathematics Assessment, National Assessment of Educational Progress (NAEP), U.S. Department of Education (Institute of Education Sciences, National Center for Education Statistics, https://www.nationsreportcard.gov/reports/mathematics/2024/g4_8/.

7   Ivana Saric, "U.S. students' math scores plunge in global education assessment," *Axios*, December 5, 2023, https://www.axios.com/2023/12/05/us-students-pisa-global-assessment; Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B., TIMSS 2019 International Results in Mathematics and Science, 2020,https://timss2019.org/reports/index.html; *2022 Results (Volume I): The State of Learning and Equity in Education*, OECD (Paris: OECD Publishing, 2023), https://www.oecd.org/en/publications/pisa-2022-results-volume-i_53f23881-en.html.

8   Education Expenditures by Country, National Center for Education Statistics, https://nces.ed.gov/programs/coe/indicator/cmd/education-expenditures-by-country.

9   Manil Suri, "Declines in math readiness underscore the urgency of math awareness," *UMBC Magazine*, March 31, 2023, https://umbc.edu/stories/math-awareness-needed-to-raise-math-readiness/.

10   The State of U.S. Science and Engineering 2024, National Science Board, https://ncses.nsf.gov/pubs/nsb20243/table/1.

alarming.[11] Tellingly, teachers fleeing the profession often list the complexity, ambiguity, and infeasibility of the educational standards and the standardized tests that accompany them as one of their primary reasons for leaving the profession.[12] Our current education standards don't just inhibit student learning; they positively drive talented teachers out of our classrooms.

In short, the United States spends an extraordinary amount of money on mathematics education to achieve mediocre results. We are falling behind our peers and rivals in math education, failure that threatens our national security as well as our prosperity. The CCSSM and its related adaptations in the states have lowered the bar too far, sacrificing both achievement and accessibility in a misguided quest for equity and inclusion.

This bleak state of affairs cries out for reform. America needs a new set of mathematics standards that addresses the issues that plague our educational institutions. These new standards should prescribe a pathway to success that emphasizes content-rich, research-based instruction that is easy for teachers to understand and implement. The *Archimedes Standards* were written in pursuit of this goal. They eliminate superfluous filler, disentangle elaborate complexities, streamline learning processes, and re-center instruction on the foundational concepts of mathematics. They are lucid, rigorous, and accessible, and they help ensure that all American students have the opportunity to learn mathematics for their own advancement and for the public good.

Archimedes Standards: Characteristics

Lucidity

The *Archimedes Standards*' central characteristic is lucidity. Other standards confuse teachers and administrators, to say nothing of parents and students, with vague and incoherent language that relies on educational jargon and entangles dissimilar mathematical concepts. Consider, for example, this Grade 2 standard from the CCSSM:

> 2.NBTB.7.   Add and subtract within 100, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

This standard complicates the simple task of adding and subtracting three-digit numbers with sixty-seven words enumerating "sometimes"-used methods, "and/or" strategies, and recommendations to use a set of unknown "strategies" and "properties." Such a muddled standard creates unnecessary confusion for teachers attempting to understand and implement the standard

11   Richard Ingersoll, et al. "The demographic transformation of the teaching force in the United States."*Education Sciences* 11.5 (2021): 234.

12   Anthony Gary Dworkin and Pamela F. Tobe, "The Effects of Standards Based School Accountability on Teacher Burnout and Trust Relationships: A Longitudinal Analysis," in *Trust and School Life*, eds. Demitri Van Maele, Patrick B. Forsyth, and Mieke Van Houtte (Dordrecht, NL: Springer Dordrecht, 2014), 121–43. https://doi.org/10.1007/978-94-017-8014-8_6.

and renders it nearly impossible for parents to hold teachers accountable for their children's math instruction.

The *Archimedes Standards*, by contrast, offer brief, clear, coherent standards that teachers, students, and parents can understand, implement, and assess. Consider the corresponding standard from the *Archimedes Standards*:

> 3.1   Add three-digit whole numbers to 1,000 with regrouping using the standard algorithm.
>
> 3.2   Subtract three-digit whole numbers within 1,000 with regrouping using the standard algorithm.

Here, the core content of addition and subtraction is centered along with a few key problem-solving methods (regrouping, the standard algorithm) that elementary teachers will be familiar with. Such lucidity and directness pinpoint the exact content knowledge to be taught in the classroom, facilitating the work of the teacher and providing flexibility in implementation across a variety of curriculum frameworks.

Practicality

The *Archimedes Standards* complement lucidity with practicality. Math standards should not place a burden upon teachers; rather, they should serve as a guide to instruction that can help teachers to plan and execute dynamic lessons. Many of the decisions throughout the writing of the *Archimedes Standards* were made with teachers in mind: stylistic simplicity, clear definitions of key terms, embedded mathematical formulas and symbols, and grade-level-appropriate vocabulary. No longer must teachers navigate *around* the standards in order to teach; rather, teachers may navigate *by* them, using each standard as a clearly articulated learning goal that functions as the centerpiece of a lesson.

Consider, for example, these standards from the *Archimedes Standards'* course in Algebra I:

> 6.1   Understand a polynomial to be a sum of algebraic terms having variables, coefficients, exponents, and/or constants.
>
> 6.2   Classify polynomials by degree and number of terms.
>
> 6.3   Add, subtract, and multiply polynomials.
>
> 6.4   Divide polynomials by monomials.

The definition of a polynomial is clear (both in the standards themselves and in the appended glossary), but not overly technical. The skills the students need to develop are well-defined and accessible, but not so narrowly prescriptive as to micromanage the teacher or eliminate the art of teaching. And the progression from standard to standard is natural and achievable, without rushing the teacher or overwhelming the student. Such standards are practical, coherent, user-friendly guides to help teachers to teach.

Content-Rich Rigor

The *Archimedes Standards* restore content-rich rigor to mathematics education. Its detailed scope and sequence lays out a pathway from Pre-K through 8th grade that will prepare students for the challenges of Algebra I when they enter 9th grade (if not sooner). This sequence will allow all students to complete Pre-Calculus by the time they graduate high school, and set them up for success in college or in their chosen field.

All too often, contemporary education standards decenter the content they ought to be focusing on. Consider this CCSSM standard on fractions from the 4th grade:

> 4.NF.A.1.   Explain why a fraction a/b is equivalent to a fraction (n x a)/(n x b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

In this standard, not only has the content has become secondary to the teaching methods, but one of the most important pieces of content—reducing fractions to their simplest form—has been lost. Even if the methods specified in this standard are appropriate, the decentering of content has compromised the math itself, encouraging holes to appear in students' understanding of important concepts. Compare this to the *Archimedes Standards'* approach to the same topic:

> 2.7   Find equivalent fractions and reduce fractions to their simplest form.

When the content is given its proper place at the center of the standards, the rigor of the discipline is restored, and students have the opportunity to develop a thorough understanding of mathematics.

With such content-rich rigor, the *Archimedes Standards* provide the foundation for an entire apparatus of curricula, textbooks, curriculum frameworks, and professional development, facilitating state and district assessment of mathematical knowledge. Such attention to detail makes it clear what exactly teachers should be expected to know, laying the groundwork for teacher preparation and professional development. All these features are geared towards producing students who graduate from high school with deep mathematical knowledge and a love of learning.

Democratic Accessibility

At the same time, the *Archimedes Standards* balance a rigorous, content-rich scope and sequence with simplicity and directness, promoting accessibility and ensuring that America's schools fulfill the promise of equal educational opportunities for everyone.

The *Archimedes Standards* have been crafted in a streamlined format that strips out unnecessary obstacles in order to promote democratic accessibility. Consider this 2nd grade standard from the CCSSM:

> 2.MD.B.5.   Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

The CCSSM's cluttered language attaches a series of tangential ideas—equations, variables, drawings, word problems, etc.—to the core idea of measurement. This places a further obstacle to learning for the students who need it most. Compare the CCSSM's verbose standards with the *Archimedes Standards'* equivalent:

> 7.2   Measure length using inches, feet, centimeters, and meters.

The *Archimedes Standards'* simplicity centers the concept of measurement, encouraging teachers to focus on the fundamentals of math knowledge and giving students direct access to the most necessary concepts and skills.

The *Archimedes Standards* were crafted to serve all American students. They focus on core content that is grade-level appropriate to ensure that students with different capabilities all learn the fundamentals of mathematics.[13] By minimizing superfluity and targeting the essentials, the *Archimedes Standards* streamline the learning process for students, freeing them to focus their time and effort on the most important aspects of mathematics.

Various states and school districts rightly want to adapt standards to suit their own students. *The Archimedes Standards* are model standards that can and should be adapted to facilitate different pathways for different groups of students pursuing different educational goals. As such, the *Standards* were crafted so that states and schools could extract material easily, promoting flexibility and facilitating local adaptation. They are standards for *all* Americans.

Depoliticized

In contrast to some of the recent efforts to revise PreK-12 math standards, the *Archimedes Standards* are staunchly apolitical, devoted entirely to the best math content. No ideologies permeate the problems and tasks, no ethnomathematics guides the pedagogies, and no tangential applications have been inserted that politicize math (e.g., using the Triangular Trade to teach students about triangles).[14] The *Archimedes Standards* were created in part as an antidote to the politicization that threatens to invade the world of math education. By its very nature, mathematics is a discipline that is free of cultural bias, universally true, and accessible to all. Thus, it can and should serve as a unifying discipline, providing principles that Americans of all backgrounds, cultures, and political affiliations can rally behind.

Research-Based Practices

The *Archimedes Standards* build upon the best practices of the past and the present, drawing upon insights from national and international sources. Excellent mathematical practices abroad (such as those in Singapore and South Korea) and within the United States (such as Florida's 2021 standards) informed the crafting of the standards and will inform the development of the supporting frameworks and curricula. A few of these research-based practices are listed below:

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13   Notable works such as Thomas Sowell's *Black Education: Myths and Tragedies* (1972), E. D. Hirsch's *The Schools We Need and Why We Don't Have Them* (1996), Annette Lareau's *Unequal Childhoods: Class, Race, and Family Life* (2003), and Abigail and Stephen Thernstrom's *No Excuses: Closing the Racial Gap in Learning* (2004) point out that disadvantaged students need intensive content instruction the most.

14   Triangular Trade Mathematics Cross-Curriculum Activity, Teachers Pay Teachers, https://www.teacherspayteachers.com/Product/Triangular-Trade-Mathematics-Cross-Curriculum-Activity-3619236.

Repetition and Spiraling: Repeated instruction in successive grades at increasing levels of complexity — called "spiraling" — greatly increases student learning.[15] In a spiral set of standards, learning becomes spread out over time as students have the chance to deepen their understanding of a concept from grade to grade. In the *Archimedes Standards,* the learning goals associated with a given concept are often stretched across multiple grades, providing students the opportunity to return multiple times to recall, review, and extend their knowledge.

Connecting Multiple Representations: The theory of multiple representations refers to the idea that many math concepts can be presented and understood through various forms like graphs, tables, equations, words, physical manipulatives, diagrams, and drawings.[16] Such a variety of access points allows students to encounter an idea from different perspectives, promoting deeper and more flexible understanding. Often, in the *Archimedes Standards,* students are required to represent a certain concept (such as a function) in multiple ways, allowing them to see the concept take shape in multiple forms.

Concrete-Pictorial-Abstract (CPA) Approach: One of the primary challenges in learning math, especially in the younger grades, is abstraction. That is, students must perform a translation of sorts, taking information provided in a real scenario and write a math sentence using symbols and numbers. The CPA approach effectively uses pictures and diagrams to mediate the concrete and the abstract, providing students with a streamlined entryway to the symbolic world of mathematics.[17] Utilizing this CPA approach, the *Archimedes Standards* recommend a series of physical tools a student could use, including linking cubes, pattern blocks, fraction circles, and base ten place value blocks, along with a series of pictures that students might draw including number bonds, place value charts, and bar models.

Supporting Procedural Fluency and Conceptual Understanding: Mathematical proficiency requires the ability to perform operations accurately and efficiently (procedural fluency) and the ability to grasp the underlying concepts and reasoning behind those operations (conceptual understanding). That is, students must understand not only "how" to do the math but also "why" the math works the way it does. These two understandings should be developed in tandem, with the conceptual understanding often laying the foundation for building procedural fluency.[18] In the *Archimedes Standards,* students are often asked first to "understand" a given concept before they then go on to use that concept in various operations.

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15   Jerome Bruner, *The Process of Education* (Boston: Harvard University Press, 1976),

16   Preety N. Tripathi, "Developing Mathematical Understanding through Multiple Representations," *Mathematics Teaching in the Middle School* 13, no. 8 (2008): 438-445, https://doi.org/10.5951/MTMS.13.8.0438.

17   Yew Hoong Leong, Weng Kin Ho, and Lu Pien Cheng, "Concrete-Pictorial-Abstract: Surveying its origins and charting its future," *The Mathematics Educator* 16, no. 1 (2015): 1-18, http://math.nie.edu.sg/ame/matheduc/tme/tmeV16_1/TME16_1.pdf.

18   Bethany Rittle-Johnson, Robert S. Siegler, and Martha Wagner Alibali, "Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process.," *Journal of Educational Psychology* 93, no. 2 (2001): 346–62, https://doi.org/10.1037//0022-0663.93.2.346.

Accessibility for Parents

The *Archimedes Standards* are designed to be readily comprehensible for parents, helping them to understand the learning objectives of their children's math classes. Recognizing the vital role of parental involvement in student success, these standards avoid instructional complexities in favor of transparent, logically-sequenced progressions. The emphasis on proven mathematical principles and the standard algorithms reinforces widely understood concepts, which also will make it easier for parents to support mathematical learning at home. Parents can use the lucid *Archimedes Standards* to hold schools and teachers accountable, foster student achievement, and encourage a learning partnership between the school and the home.

Archimedes Standards: Structure

Scope and Sequence

The *Archimedes Standards* have been comprehensively designed to cover the essential topics of mathematics, while remaining adaptable to the needs of particular schools, districts, and states. While not all states and school districts provide Pre-Kindergarten and not all students will go on to Grade 12 math, the *Archimedes Standards* describe instruction for every grade from Pre-Kindergarten to Grade 12, which states and school districts may select from appropriately (see Scope and Sequence Map).

In the PreK-Grade 8 grade bands, the standards are grade-level-appropriate and build a strong framework for students' math proficiency to grow grade-by-grade. Though the recommended scope and sequence is relatively inflexible in these grades, such rigidity is essential to ensuring a solid foundation for all students. At the high school level, however, the standards divide into course sequences—Algebra I, Geometry, Algebra II, Pre-Calculus, Calculus, and Statistics—rather than grade bands. Schools may teach these six courses in whatever sequence they think appropriate, and may even divide instruction within grades. Such a combination of specific instruction for PreK-Grade 8 and maximum flexibility throughout high school, will best equip students for mathematical fluency and success.

A special note about the timing of the Algebra I class: many education reformers have argued strongly that Algebra I needs to be taught in Grade 8 rather than Grade 9 to ensure that students have enough instruction time in high school to be properly prepared for undergraduate mathematics.[19] While a proper set of standards should allow students to take Algebra I in Grade 8, not every student will be ready for such a pace. The *Archimedes Standards'* middle school sequence therefore allows for both regular and accelerated instruction. While the standards schedule Algebra I for Grade 9, schools can consolidate the Grade 6-8 material into two grades, allowing students to proceed to Algebra I a year early in Grade 8. The specific details of such a consolidation will vary from school to school, but middle school math teachers will see that the standards facilitate accelerated instruction.

This flexible approach to Algebra I exemplifies how the *Archimedes Standards* aim for both achievement and accessibility. To prescribe Algebra I for either Grade 8 or Grade 9 would be too

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19   David Randall, "Science and Math Teaching Need an Upgrade," *Real Clear Education*, January 3, 2023, https://www.realcleareducation.com/articles/2023/01/03/science_and_math_teaching_need_an_upgrade_110805.html.

narrow; leaving open the opportunity for students to take it in either grade ensures that *all* students will receive the best possible mathematics education.

Another special note about the Calculus and Statistics standards: the standards were designed to provide rigorous math education for all students rather than encouraging schools to skim hastily toward advanced placement classes. However, at the end of such a robust progression through mathematics, many high school students will be prepared to take advanced placement courses, and thus the Calculus and Statistics standards are approximately equivalent in content and rigor to AP Calculus BC and AP Statistics. While many schools will not require these courses, good standards are available for the schools that offer them.

In-Grade Sequencing

Within each grade band, the individual topics are arranged thematically (for intellectual coherence) and pedagogically (to guide the order in which the material is taught). In the rare cases in which the thematic and the pedagogical conflict, priority has been given to thematic coherence to make the standards clearer for the reader.

Consider, for example the following standards from the 5th grade:

> 1.4   Write a whole number as a product of prime factors using exponents to show factor multiples.
>
> 4.1   Understand exponents as notation describing repeated multiplication of the same number.

The standard on prime factorization fits thematically with the rest of the standards on whole numbers and thus has been included in the first 5th grade standard. Pedagogically, however, it belongs after students have been introduced to exponents, requiring teachers to reorder their instruction appropriately. Whenever possible, however, the standards do follow a pedagogical order and thus provide a rough guide for sequencing lesson plans throughout a math class.

Repeated Standards

Often a single concept (such as the slope of a line) is important enough to be covered in multiple successive years. In these cases, the standard itself is repeated word for word in all the applicable grades so that each teacher emphasizes the appropriate concepts. To understand the prior knowledge their students will possess and to see the kind of preparation their students will need for the grades to come, teachers should glance at the standards for the year before and the year after the grade level that they teach. Such a holistic perspective will help them to determine if a concept needs to be repeated or reviewed.

Mental Mathematics

In contrast to the CCSSM, which follows the modern trend of deemphasizing math facts, the *Archimedes Standards* are committed to encouraging students to practice mental math in order to develop number sense and number flexibility. While mathematics is fundamentally a creative endeavor that cannot be reduced to a set of memorizable processes, math instruction cannot be divorced

from the art of memory. Memorizing facts, processes, and formulas is useful for daily life, equipping students with the tools to quickly and efficiently solve practical problems at home, at work, and at play. Mental math also greatly facilitates the ability to engage in higher-level mathematics, providing a solid foundation from which to explore advanced concepts. Practically speaking, a true preparation for undergraduate and professional mathematics requires early memorization, which should remain central to math instruction.

Throughout the K-6 sequence, each grade level includes several Mental Mathematics standards, each of which specify mathematical knowledge that teachers should encourage students to commit to memory. Individual districts, schools, and teachers should choose the specific methods of doing so (flashcards, games, etc.), but every classroom should develop a culture of mental math that helps make mathematical reasoning second nature for students.

Mathematical Practices

The *Archimedes Standards'* emphasis on content-rich knowledge contrasts with pedagogical approaches, such as "inquiry-based learning." While not bad in and of themselves, such pedagogies too often encourage teachers to replace robust instruction in content with hollow instruction in "skills." Yet the *Archimedes Standards* do not mandate or forbid specific instructional methods. As a result, teachers may use the standards to choose freely from appropriate content-focused pedagogies that will promote student learning. State education departments, likewise, are equally free to provide pedagogy guides for teachers.

Centering mathematical content, however, does not mean ignoring the specific skills that students should learn at various levels, skills which are included in the Mathematical Practices section for each grade. These practices are associated with a variety of pedagogical techniques, so that teachers can employ the particular instructional methods that they see fit. Skills instruction does have an appropriate time and place in math teaching and learning, and the Mathematical Practices section provides a helpful guide to teachers without diminishing the primary task of teaching content.

To that end, the Mathematical Practices section is limited to the following six core practices present in each grade band, outlining the kinds of activities students might undertake:

> Solve word problems using mathematical concepts.
> Use appropriate physical and conceptual tools.
> Understand and use appropriate mathematical language and terminology.
> Create representations of mathematical scenarios, problems, and processes.
> Reason about mathematical relationships and give evidence for conclusions.
> Communicate mathematical thinking.

These six practices provide a foundation for the teaching of mathematical skills, encouraging students to think mathematically without compromising content-rich instruction.

Interdisciplinary Integration

While remaining centered around content-rich math instruction, the *Archimedes Standards* have an interdisciplinary ambition that complements instruction in science, social studies, and English

language arts. This integrated, liberal-arts approach to learning helps students to make connections between different academic subjects, providing a well-rounded approach to mathematics. As such, the *Archimedes Standards* complement the two previous model standards drafted by the National Association of Scholars (NAS) and Freedom in Education (FiE), *The Franklin Standards* (science) and *American Birthright* (social studies) as well as future model standards for English language arts.[20] But more broadly, such an integrated approach will help students to see how math overlaps with other disciplines and extends into our daily lives.

History of Mathematics

One concrete expression of the commitment to interdisciplinarity is the History of Mathematics standards present in each grade and course. These standards encourage teachers and students to encounter the stories of mathematical discoveries, to walk in the shoes of great mathematicians, and to explore the nature and purpose of math. The *Archimedes Standards* incorporate these sections in moderation: not all schools and teachers will have the classroom time to incorporate these sections, and they easily be reduced or even eliminated. Still, for states, districts, and schools that do adopt them, these sections can provide an instructive and stimulating access point for students to encounter the narrative side of math.

The History of Mathematics standards have been crafted to include stories and themes that complement the math being taught at each specific grade level. In Grade 5, for example, Brahmagupta's discovery of negative numbers accompanies classroom instruction on negative numbers. Stories like this can spark student interest and help them participate in the process of guided reinvention as they investigate the origins of mathematical concepts.[21]

Archimedes Standards: Notes

Terminology

Several terms appear throughout the standards in particular ways, which readers should note.

> *Understand*: Most sections begin with an "understand" statement. These are *definitional* standards that aim at the conceptual understanding of a key idea or term. It is intended that teachers give the students the chance to explore these core mathematical concepts, discovering their meaning and importance.

> *Know that*: Some standards begin with "know that." These standards specify certain *facts* that students ought to know, particularly as prerequisite knowledge for other standards within the section, but they do not necessarily need to be accompanied by a mathematical proof or derivation.

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20 *The Franklin Standards: Model K-12 State Science Standards* (National Association of Scholars, 2024), https://www.nas.org/reports/the-franklin-standards; *American Birthright: The Civics Alliance's Model K-12 Social Studies Standards* (National Association of Scholars, 2022), https://civicsalliance.org/american-birthright/.

21 Adrian Treffers A., 1987. Three dimensions. A Model of Goal and Theory Description in Mathematics Instruction - The Wiskobas Project, D. Reidel Publ. Co., Dordrecht

Derive: Other standards begin with "derive" and go on to specify a *formula* or algorithm that students should learn. While a comprehensive proof is not always recommended, students should at least participate in the discovery of these formulas and should always be able to explain why they are true.

As an example of how these terms work together, consider the following set of standards from sixth grade:

12.1 Understand π (pi) to be the constant ratio of a circle's circumference to its diameter.
12.2 Know and use numerical approximations for π, including 3.14 and 22/7.
12.3 Derive and use formulas for the circumference ($C = 2\pi r$) and area ($A = \pi r^2$) of a circle, where r is the circle's radius.

This section first encourages students to explore the idea of π conceptually, exploring circles and discovering that no matter the size of the circle, the diameter wraps around the circumference just over three times. From that conceptual exploration, students must then know a set of facts about π, namely its numerical approximations, and must use those facts to perform calculations using π, namely the finding of radii, diameters, and circumferences. Finally, students are encouraged to derive the mathematical formulas that tie these concepts and facts together. Here, students would not be expected to provide a formal argument or proof of these formulas, but they should know where the formulas come from and why they make sense.

Standard Algorithms

The standard algorithms for the operations of addition, subtraction, multiplication, and division are important for efficiency in calculation, mathematical understanding, and preparation for algebra and beyond.[22] The *Archimedes Standards* reflect this importance by including the phrase "using the standard algorithm" where appropriate in the 2nd through 5th grade standards.

While the standard algorithms are sometimes misunderstood as mechanical procedures that dull mathematical creativity and reasoning, these algorithms use and reinforce a broad array of prior conceptual and procedural knowledge. The use of any algorithm, at every applicable grade level, should be accompanied by an understanding of how the algorithm works, not just of what it accomplishes. Pictures of the standard algorithm for the four major arithmetic operations are included below.

Addition	Subtraction	Multiplication	Division

Challenge Standards

In the 9th-12th grade bands, there are several standards marked as "challenge" topics. While these standards remain content-rich and grade-level appropriate, they are tangential to the course's primary body of knowledge. Teachers should consider ways to incorporate these standards into their lessons, but they should also feel free to omit them if appropriate.

Calculators

The use of calculators in the classroom is a particularly thorny issue. On the one hand, harnessing the power of technology to complete monotonous and difficult calculations can eliminate drudgery and tedium and free students' time and energy to focus on thinking and problem solving. On the other hand, relying on calculators to complete mathematical tasks can reduce conceptual understanding and reinforce students' beliefs that math is merely a set of 'magical' processes that somehow produce an answer in the end.

The *Archimedes Standards* chart a middle path. The PreK-8 grade standards specify calculator-free instruction, preserving a time period for developing conceptual understanding and procedural fluency. In these grades, each and every standard can and should be done "by hand." These accomplishments will help students succeed at the higher grade levels, where calculators and other technology can facilitate the exploration of complex ideas. Students should use a scientific (non- graphing) calculator in both the Algebra I and Geometry classes, especially to explore logarithmic and trigonometric functions. Students then should use a graphing utility calculator in the Algebra II, Pre-Calculus, Calculus, and Statistics classes to graph and analyze all the various classes of functions. Learning how to use calculators effectively is a key component of a PreK-12 math sequence, but it must be delayed until the appropriate moment lest it rob students of important developmental opportunities.

Conclusion

The *Archimedes Standards* facilitate excellent mathematics instruction, helping teachers provide the kind of content-rich math education that students need to become free, productive, and wise citizens. Where the Common Core State Standards for Mathematics and its various derivatives have trapped teachers and students in a cycle of mathematical mediocrity, the *Archimedes Standards*, modified to fit local situations in each state and school district, provide a distinct alternative. Under the guidelines prescribed by the *Archimedes Standards*, the United States can reclaim its heritage as a global leader in the field of mathematics, and all American students can be well-equipped with a set of mathematical tools that will aid their future endeavors.

Scope and Sequence Map

		PK	K	1	2	3	4	5	6	7	8
NUMBER	Name and count numbers	■	■	■	■	■	■				
	Understand and use whole numbers	■	■	■	■	■	■	■			
	Identify and describe fractions			■	■	■	■	■	■		
	Identify and describe decimals						■	■	■		
	Identify and describe exponents and roots								■	■	■
	Identify and describe ratios and rates							■	■	■	
	Identify and describe percentages and proportions							■	■	■	
	Understand and use negative numbers								■	■	■
	Understand and use rational numbers								■	■	■
	Understand and use irrational numbers									■	■
	Understand and use real numbers										■
ARITHMETIC	Add and subtract whole numbers	■	■	■	■	■	■				
	Multiply and divide whole numbers			■	■	■	■	■			
	Add and subtract fractions					■	■	■	■		
	Multiply and divide fractions							■	■		
	Add, subtract, multiply, and divide decimals							■	■		
	Add, subtract, multiply, and divide negative numbers								■		
GEOMETRY	Identify, describe, and draw common shapes	■	■	■	■	■	■	■	■	■	■
	Find the area and perimeter of shapes						■	■	■	■	
	Find the volume and surface area of shapes								■	■	■
	Understand and use x-y coordinate plane								■	■	■
	Understand and use rigid transformations										■

		PK	K	1	2	3	4	5	6	7	8
MEASUREMENT	Sort and classify objects	■									
	Measure and compare objects using various units	■	■	■	■	■	■	■	■		
	Understand time and money	■	■	■	■	■					
PRE-ALGEBRA	Understand variables and formulas								■		
	Simplify and evaluate expressions								■	■	■
	Solve one-variable equations								■	■	■
	Solve one-variable inequalities									■	■
	Understand and represent two variable equations									■	■
	Understand and use polynomials										■
	Understand and represent linear equations										■
	Solve systems of linear equations										■
	Identify and describe functions										■
STATISTICS AND PROBABILITY	Represent data using graph or chart		■	■	■	■	■	■	■		
	Analyze data using statistical measures.							■	■	■	■
	Understand probability									■	■
MENTAL MATHEMATICS			■	■	■	■	■	■	■		
MATHEMATICAL PRACTICES		■	■	■	■	■	■	■	■	■	■
HISTORY OF MATHEMATICS		■	■	■	■	■	■	■	■	■	■

The Model Pre-

K-12 Standards

Pre-Kindergarten

Numbers

1. Count to 20.
   - 1.1 Count by ones to 20.
   - 1.2 Given a number between 1 and 20, state the preceding and following numbers.

2. Understand and use numbers to 10.
   - 2.1 Identify and read numbers to 10 using numerals.
   - 2.2 Count the number of objects in a group (one-to-one correspondence).
   - 2.3 Represent the total number of objects in a group with the appropriate spoken numeral (cardinality).
   - 2.4 Given a number to 10, count out that many objects.
   - 2.5 Compare the number of objects in two groups using the words more than, less than, or same.
   - 2.6 Represent a count of no objects with the number 0.

Arithmetic

3. Explore addition and subtraction.
   - 3.1 Put together two groups of objects and say the total (up to 5).
   - 3.2 Take away some objects from a group (up to 5) and say how many are left.

Geometry

4. Identify common shapes.
   - 4.1 Identify circles, triangles, squares, and rectangles in a variety of orientations.

Measurement

5. Match, sort, and classify objects.
    5.1   Sort objects into two groups based upon a characteristic, such as color, texture, size, or function.
    5.2   Identify objects that do not belong in a particular group.
    5.3   Identify and continue simple patterns of numbers, shapes, or objects.

6. Compare objects using appropriate measurement vocabulary.
    6.1   Compare objects by size using the words large/big and small.
    6.2   Compare objects by weight using the words heavy and light.
    6.3   Compare objects by length and height using the words long, tall, and short.
    6.4   Compare objects by position using the words first, second, third, fourth, and fifth.

7. Explore time and money.
    7.1   Describe time with words, including day, night, today, yesterday, tomorrow, minute, hour, week, month, and year.
    7.2   Identify the tools that measure time, including clocks and calendars.
    7.3   Name the days of the week.
    7.4   Identify U.S. coins by name.

Mathematical Practices

8. Students should:
    8.1   *Solve word problems using mathematical concepts including*: counting objects, putting together groups of objects, taking away objects from a group, sorting objects, comparing objects, and time.
    8.2   *Use appropriate physical and conceptual tools including*: linking cubes, ten frames, geometric pattern blocks, number cards, clocks, and calendars.
    8.3   Understand and use appropriate mathematical language and terminology including: number, count, more, less, same, part, whole, group, shape, and measure.
    8.4   Create representations of mathematical scenarios, problems, and processes including: pictures and sketches.
    8.5   Reason about mathematical relationships and give evidence for conclusions by: counting, comparing, sorting, and naming.
    8.6   *Communicate mathematical thinking by*: telling stories about a collection of objects, telling stories using appropriate language to describe time, and explaining why they gave a particular answer.

History of Mathematics

9.   Listen to true stories of famous mathematicians. Such as:
- 9.1   The Ishango Bone, the act of counting, and primitive mathematics.
- 9.2   Leonardo Da Vinci and his flying machines and other mathematical inventions.

10. Explore the nature and purpose of mathematics.
- 10.1   Describe reasons why we use numbers to understand the world around us.
- 10.2   Connect mathematics to daily life by noticing the various kinds of objects that can be counted.
- 10.3   Imagine some of the problems that mathematics can help to solve.

Kindergarten

Numbers

1. Count to 100.
    1.1   Count by ones, twos, fives, and tens to 100.
    1.2   Count forwards or backwards by ones and by tens from a starting number.

2. Understand and use numbers to 30.
    2.1   Read and write numbers to 30 using numerals.
    2.2   Count the number of objects in a group (one-to-one correspondence).
    2.3   Represent the total number of objects in a group with the appropriate numeral (cardinality).
    2.4   Given a number to 30, count out that many objects.
    2.5   Compare the number of objects in two groups using the words less than, greater than, or equal to.
    2.6   Arrange a group of whole numbers to 30 in increasing or decreasing order.
    2.7   Place whole numbers to 30 on a number line.
    2.8   Break apart and put together numbers to 30 into tens and ones.
    2.9   Separate sets of objects into equal groups.
    2.10   Represent a count of no objects with the number 0.

Arithmetic

3. Add and subtract whole numbers to 10.
    3.1   Understand *addition* as putting together, getting more, and counting on.
    3.2   Add one-digit whole numbers to 10.
    3.3   Understand *subtraction* as taking away, taking apart, and counting back.
    3.4   Subtract one-digit whole numbers within 10.
    3.5   Represent the operations of addition and subtraction with the symbols + and -.
    3.6   Represent the results of addition and subtraction with the symbol =.
    3.7   Find number pairs that add to 10.
    3.8   Find the different ways that a whole number (from 2 to 10) can be broken into two parts.

Geometry

4. Identify, describe, compare, and draw common shapes.
    4.1 Understand a *triangle* to be a flat shape with three sides.
    4.2 Understand a *rectangle* to be a flat shape with four sides and L-shaped corners.
    4.3 Understand a *square* to be a rectangle where all the sides are the same length.
    4.4 Understand a *circle* to be a flat shape that is perfectly round.
    4.5 Identify and draw circles, triangles, squares, and rectangles in a variety of orientations.
    4.6 Divide two-dimensional shapes into equal parts.

Measurement

5. Compare and sort objects using appropriate measurement vocabulary.
    5.1 Compare and sort objects by size using the words larger and smaller.
    5.2 Compare and sort objects by weight using the words heavier and lighter.
    5.3 Compare and sort objects by length using the words longer, shorter, taller, higher, and lower.
    5.4 Compare and sort objects by capacity using the words holds more and holds less.
    5.5 Compare and sort objects by position using the words first, second, third, fourth, fifth, and so on.
    5.6 Compare and sort objects by temperature using the words warmer, cooler, hotter, and colder.
    5.7 Compare and sort objects by location using the words inside, outside, between, above, below, left of, right of, and next to.
    5.8 Identify objects that do not belong in a particular group.

6. Understand time and money.
    6.1 Describe time with words, including morning, afternoon, evening, today, yesterday, tomorrow, minute, hour, day, week, month, and year.
    6.2 Know and use the tools that measure time, including clocks and calendars.
    6.3 Name the days of the week and the months of the year.
    6.4 Tell time to the hour on analog and digital clocks.
    6.5 Identify U.S. coins and bills by name and tell their values.
    6.6 Identify the dollar sign ($) and the cents sign (¢).

7. Represent and analyze data.
    7.1 Collect and sort objects into categories and compare the categories using a picture graph.

Mental Mathematics

8. Memorize or mentally calculate math facts to improve mental mathematics.
    8.1   Recall from memory addition facts up to sum 10 and the corresponding subtraction facts.
    8.2   Mentally calculate the number that is 1 more or 1 less than a given number.

Mathematical Practices

9. Students should:
    9.1   *Solve word problems using mathematical concepts including*: groups of objects, addition, subtraction, measurement, comparison, and time.
    9.2   *Use appropriate physical and conceptual tools including*: linking cubes, geometric pattern blocks, counting chips, ten frames, number cards, clocks, and calendars.
    9.3   *Understand and use appropriate mathematical language and terminology including*: greater than, less than, equal to, part, whole, group, measure, and appropriate comparative vocabulary.
    9.4   Create representations of mathematical scenarios, problems, and processes including: pictures, sketches, number bonds, and number lines.
    9.5   *Reason about mathematical relationships and give evidence for conclusions by*: explaining the concepts of addition and subtraction, describing the differences between two or more objects, and identifying patterns in numbers, shapes, sizes, objects, or colors.
    9.6   *Communicate mathematical thinking by*: asking and answering "how many?" questions, telling addition and subtraction stories about a collection of objects, identifying parts and wholes, and explaining why they gave a particular answer.

History of Mathematics

10. Listen to true stories of famous mathematicians. Such as:
    10.1   Sumerian tablets and the use of arithmetic as early as 2600 BC.
    10.2   Thomas Jefferson and the necessity of mathematics for good citizenship.

11. Explore the nature and purpose of mathematics.
    11.1   Describe ways in which mathematics is similar to yet different from other subjects.
    11.2   Connect mathematics to game playing by investigating the role of mathematics in creating, playing, and winning a game.
    11.3   Imagine the origins of mathematics and some of the questions the earliest mathematicians might have encountered.

Grade One

Numbers

1. Understand and use whole numbers to 100.
   1.1 Count, read, and write whole numbers to 100 using numerals and words.
   1.2 Identify the place value of each digit for whole numbers to 100.
   1.3 Compare whole numbers to 100 using the symbols <, >, and =.
   1.4 Arrange a group of whole numbers to 100 in increasing or decreasing order.
   1.5 Place whole numbers to 100 on a number line.
   1.6 Break apart and put together whole numbers to 100 into tens and ones.
   1.7 Identify whole numbers to 100 as even or odd.

2. Explore fractions.
   2.1 Break circles and rectangles into two, three, and four equal-sized parts.
   2.2 Identify and describe halves, thirds, and fourths given a visual representation.

Arithmetic

3. Add and subtract whole numbers to 100.
   3.1 Add one and two-digit whole numbers to 100 with regrouping.
   3.2 Subtract one and two-digit whole numbers to 100 with regrouping.
   3.3 Solve comparison problems with addition and subtraction.
   3.4 Know that addition and subtraction are opposite operations (inverse operations).
   3.5 Find unknown values in addition and subtraction equations.

4. Explore operations with equal groups.
   4.1 Skip count by different whole numbers to 100, including 2s, 3s, 4s, 5s, and 10s.
   4.2 Add equal groups of one and two-digit numbers.
   4.3 Split one and two-digit numbers into equal parts (fair shares).

Geometry

5. Identify, describe, compare, and draw common shapes.
    5.1   Understand a *semi-circle* to be half of a circle.
    5.2   Understand a *pentagon* to be a flat shape with five sides.
    5.3   Understand a *hexagon* to be a flat shape with six sides.
    5.4   Identify and draw circles, semi-circles, triangles, squares, rectangles, pentagons, and hexagons in a variety of orientations.
    5.5   Describe and compare two-dimensional shapes by their basic properties.
    5.6   Break apart two-dimensional shapes, and identify and describe the resulting smaller shapes.

Measurement

6. Measure and compare objects.
    6.1   Understand *measurement* to be the repetition of a unit.
    6.2   Distinguish between standard units and non-standard units.
    6.3   Identify a ruler as an instrument to measure length.
    6.4   Measure the length of an object with non-standard units.
    6.5   Measure the length of an object to the nearest inch or centimeter.
    6.6   Identify a scale as an instrument to measure weight.
    6.7   Measure the weight of an object on a balance scale with non-standard units.
    6.8   Measure the weight of an object to the nearest pound or kilogram.
    6.9   Order objects using 1st, 2nd, 3rd, and so on (ordinal numbers).

7. Understand time and money.
    7.1   Tell time to fifteen-minute intervals on analog and digital clocks.
    7.2   Find the value of combinations of U.S. coins up to one dollar.
    7.3   Find the value of combinations of U.S. bills up to $100.
    7.4   Write money values using the symbols $ and ¢.
    7.5   Show different combinations of coins that equal the same value.

8. Represent and analyze data.
    8.1   Collect and sort objects into categories and represent the data using a picture graph or tally chart.
    8.2   Interpret and compare data represented in a picture graph or tally chart.

Mental Mathematics

9. Memorize or mentally calculate math facts to improve mental mathematics.
    9.1   Mentally calculate the number that is 1 more, 1 less, 10 more, or 10 less than a given number.
    9.2   Recall from memory addition facts (up to 10+10) and the corresponding subtraction facts.

Mathematical Practices

10. Students should:
    10.1   *Solve word problems using mathematical concepts including*: addition, subtraction, length, weight, money, and time.
    10.2   *Use appropriate physical and conceptual tools including*: linking cubes, geometric pattern blocks, ten frames, base ten place value blocks, rulers, scales, coins, bills, and clocks.
    10.3   Understand and use appropriate mathematical language and terminology including: place value, one, ten, hundred, plus, minus, groups, units, and graph.
    10.4   Create representations of mathematical scenarios, problems, and processes including: number bonds, number lines, place value charts, and picture graphs.
    10.5   *Reason about mathematical relationships and give evidence for conclusions by*: employing various methods of addition and subtraction, distinguishing between what is known and unknown in a problem, and identifying and describing numerical patterns.
    10.6   *Communicate mathematical thinking by*: telling addition and subtraction stories, writing and evaluating mathematical sentences, explaining the choice of a particular addition or subtraction strategy, and explaining why an answer does or does not make sense.

History of Mathematics

11. Read or listen to true stories of famous mathematicians. Such as:
    11.1   Chinese bamboo multiplication tables and base 10 arithmetic.
    11.2   The Dresden Codex and the Mayan base 20 numbering system.
    11.3   Archimedes and his lever.
    11.4   Eratosthenes and the measurement of the earth's circumference.

12. Explore the nature and purpose of mathematics.

12.1   Describe some of the mathematical patterns that occur in nature.

12.2   Connect mathematics to economics by investigating the ways in which money depends on mathematics.

12.3   Imagine how mathematics might be different if the world were different (e.g., if humans had 8 fingers, if the world were flat).

Grade Two

Numbers

1. Understand and use whole numbers to 1,000.
   1.1   Count, read, and write whole numbers to 1,000 using numerals and words.
   1.2   Write whole numbers to 1,000 in expanded form and write numerals for numbers expressed in expanded form.
   1.3   Identify the place value of each digit for whole numbers to 1,000.
   1.4   Compare whole numbers to 1,000 using the symbols <, >, and =.
   1.5   Arrange a group of whole numbers to 1000 in increasing or decreasing order.
   1.6   Place whole numbers to 1000 on a number line.
   1.7   Break apart and put together whole numbers to 1,000 into various combinations of hundreds, tens, and ones.
   1.8   Identify whole numbers to 1,000 as even or odd.

2. Identify and describe fractions.
   2.1   Understand *fractions* as parts of a whole.
   2.2   Understand the *denominator* as the number on the bottom of a fraction representing the number of equal-sized parts the whole is broken into.
   2.3   Understand the *numerator* as the number on the top of a fraction representing the number of selected equal-sized parts.
   2.4   Identify, name, and compare the unit fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.
   2.5   Name fractions with denominators 2, 3, and 4 using numerals and words
   2.6   Draw representations of fractions with denominators 2, 3, and 4.
   2.7   Identify fractions (such as $\frac{4}{4}$) that are equal to the whole, and know that such fractions are also equal to 1.

Arithmetic

3. Add and subtract whole numbers to 1,000.
   3.1   Add three-digit whole numbers to 1,000 with regrouping using the standard algorithm.

3.2   Subtract three-digit whole numbers within 1,000 with regrouping using the standard algorithm.

3.3   Find unknown values in addition and subtraction equations using inverse operations.

3.4   Solve problems using the commutative $(a + b = b + a)$ and associative $[a + (b + c) = (a + b) + c]$ properties of addition.

3.5   Identify, create, describe, and extend addition and subtraction number patterns.

4. Multiply and divide whole numbers within 100.

4.1   Understand *multiplication* as repeated addition and as putting together equal groups.

4.2   Multiply whole numbers within 100 (groups of 2s, 3s, 4s, 5s, and 10s).

4.3   Understand *division* as repeated subtraction and as breaking wholes into equal groups.

4.4   Divide whole numbers within 100 (groups of 2s, 3s, 4s, 5s, and 10s).

Geometry

5. Identify, describe, compare, and draw common shapes.

5.1   Understand an *octagon* to be a flat shape with eight sides.

5.2   Describe and compare circles, semi-circles, triangles, squares, rectangles, pentagons, hexagons, and octagons by their basic properties.

5.3   Distinguish between a flat, 2-dimensional shape and a solid, 3-dimensional shape.

5.4   Understand a *rectangular prism* to be a solid shape that looks like a box with six flat faces that are all rectangles.

5.5   Understand a *cube* to be a rectangular prism where all the faces are squares.

5.6   Understand a *pyramid* to be a solid shape with a flat shape at the base and a single point at the top.

5.7   Understand a *cone* to be a pyramid with a circular base.

5.8   Understand a *cylinder* to be a solid shape that looks like a can with a round tube and a circle at both ends.

5.9   Understand a *sphere* to be a perfectly round solid shape with no corners or sides.

5.10   Describe and compare spheres, cubes, cones, pyramids, rectangular prisms, and cylinders by their basic properties.

Measurement

6. Measure and compare objects.

6.1   Distinguish between English and metric units of measurement.

6.2   Measure length to the nearest unit using inches, feet, centimeters, and meters.

6.3   Measure weight to the nearest unit using ounces and pounds.

6.4   Measure mass to the nearest unit using grams and kilograms.

7. Understand time and money.
    7.1 Tell time to five-minute intervals on analog and digital clocks.
    7.2 Determine the duration of intervals of time to five-minute intervals (e.g., how much time has passed between 6:05 and 6:50).
    7.3 Know the difference between a.m. (ante meridiem) and p.m. (post meridiem).
    7.4 Represent amounts of money to $100.00 using decimal notation.

8. Represent and analyze data.
    8.1 Create tables, picture graphs, and bar graphs to represent a given set of data.
    8.2 Interpret and analyze data presented with tables, picture graphs, and bar graphs.

Mental Mathematics

9. Memorize or mentally calculate math facts to improve mental mathematics.
    9.1 Mentally calculate the number that is 1 more, 1 less, 10 more, 10 less, 100 more, or 100 less than a given number.
    9.2 Recall from memory addition facts (up to 10+10) and the corresponding subtraction facts.
    9.3 Recall from memory multiplication facts of 2s, 3s, 4s, 5s, and 10s and the corresponding division facts.
    9.4 Mentally add and subtract whole numbers to 100.

Mathematical Practices

10. Students should:
    10.1 *Solve word problems using mathematical concepts including*: addition, subtraction, multiplication, division, length, capacity, weight, time, and money.
    10.2 *Use appropriate physical and conceptual tools including*: geometric pattern blocks, base ten place value blocks, fraction strips and circles, rulers, measuring cups, scales, coins, bills, and clocks.
    10.3 *Understand and use appropriate mathematical language and terminology including*: digit, value, thousand, operations, addition, subtraction, multiplication, division, regrouping, commutative, associative, fraction, two-dimensional, three-dimensional, a.m., p.m., face, edge, and vertex.
    10.4 *Create representations of mathematical scenarios, problems, and processes including*: number bonds, arrays, place value charts, bar models, fraction strips, tables, pictographs, and bar graphs.

10.5   *Reason about mathematical relationships and give evidence for conclusions by*: estimating values, predicting answers, checking answers, describing the differences between various understandings of multiplication and division, and solving problems in multiple ways.

10.6   *Communicate mathematical thinking by*: telling addition, subtraction, multiplication, and division stories, writing and evaluating mathematical sentences, defending their choice of a particular mathematical strategy, and explaining why an answer does or does not make sense.

History of Mathematics

11.   Read or listen to true stories of famous mathematicians. Such as:

11.1   Brahmagupta and the use of the number 0.

11.2   Henry I and the definition of the yard.

11.3   Al-Khwarizmi and the development of the standard algorithm.

11.4   Plato and the five platonic solids.

12.   Explore the nature and purpose of mathematics.

12.1   Describe the qualities or traits (creativity, curiosity, perseverance, etc.) of an excellent mathematician.

12.2   Connect mathematics to science by using graphs or charts to represent scientific data.

12.3   Imagine the discovery or invention of a new mathematical idea.

Grade Three

Numbers

1. Understand and use whole numbers to 10,000.
   - 1.1 Count, read, and write whole numbers to 10,000 using numerals and words.
   - 1.2 Write whole numbers to 10,000 in expanded form and write numerals for numbers expressed in expanded form.
   - 1.3 Identify the place value of each digit for whole numbers to 10,000.
   - 1.4 Compare whole numbers to 10,000 using the symbols <, >, and =.
   - 1.5 Arrange a group of whole numbers to 10,000 in increasing or decreasing order.
   - 1.6 Place whole numbers to 10,000 on a number line.
   - 1.7 Break apart and put together whole numbers to 10,000 into various combinations of thousands, hundreds, tens, and ones.
   - 1.8 Round whole numbers to the nearest ten, hundred, or thousand.

2. Identify and describe fractions.
   - 2.1 Understand *fractions* as parts of a whole, as numbers (on a number line), and as multiples of unit fractions.
   - 2.2 Name fractions less than or equal to 1 using numerals and words.
   - 2.3 Draw representations of fractions.
   - 2.4 Compare fractions to the benchmark fractions 0, $\frac{1}{2}$, and 1.
   - 2.5 Compare fractions with the same denominators using the symbols <, >, and =.
   - 2.6 Compare fractions with the same numerator using the symbols <, >, and =.
   - 2.7 Identify equivalent fractions given visual representations.

Arithmetic

3. Add and subtract whole numbers to 10,000.
   - 3.1 Add whole numbers to 10,000 using the standard algorithm.
   - 3.2 Subtract whole numbers within 10,000 using the standard algorithm.

4. Multiply and divide whole numbers within 10,000.
    4.1   Multiply whole numbers within 10,000 (multi-digit numbers by one-digit numbers) using the standard algorithm.
    4.2   Divide whole numbers within 10,000 (multi-digit numbers by one-digit numbers) using the standard algorithm.
    4.3   Distinguish between how-many-groups (quotative/measurement) division and how-many-in-each-group (partitive/sharing) division.
    4.4   Identify multiplication and division as inverse operations.
    4.5   Find unknown values in multiplication and division equations.
    4.6   Understand and use the *commutative* $(a * b = b * a)$ and *associative* $[a*(b*c) = (a*b)*c]$ properties of multiplication.
    4.7   Know and use the properties of 0 and 1 in multiplication and division, and know that a number cannot be divided by 0.

5. Add and subtract fractions.
    5.1   Add and subtract fractions with the same denominators (like fractions).

Geometry

6. Identify, describe, compare, and draw common geometrical objects.
    6.1   Identify, describe, and draw lines, segments, rays, and angles.
    6.2   Identify, describe, and draw intersecting, parallel, and perpendicular lines.
    6.3   Identify and draw lines of symmetry.
    6.4   Identify and describe various quadrilaterals by their properties, including parallelograms, rhombuses, rectangles, squares, and trapezoids.
    6.5   Know the relationships between the various quadrilaterals (e.g. all squares are rectangles but not all rectangles are squares).

7. Find the area and perimeter of shapes.
    7.1   Understand *perimeter* as the distance around a 2-dimensional shape.
    7.2   Find the perimeter of triangles, quadrilaterals, pentagons, and hexagons.
    7.3   Understand *area* as the amount of space inside a 2-dimensional shape.
    7.4   Find the area of rectangles and squares.

Measurement

8. Measure and compare objects.
    8.1   Measure capacity (liquid volume) using cups, pints, quarts, gallons, liters, and milliliters.
    8.2   Read the temperature on a thermometer in both Celsius and Fahrenheit.

9. Understand time and money.
    9.1   Tell time to the minute on analog and digital clocks.
    9.2   Add and subtract time to the minute.
    9.3   Determine the duration of intervals of time to one-minute intervals.
    9.4   Add, subtract, multiply, and divide money to $1000.00 using decimal notation.
    9.5   Make correct change.

10. Represent and analyze data.
    10.1   Create tables, bar graphs, circle graphs, and line graphs to represent a given set of data.
    10.2   Interpret and analyze data presented with tables, bar graphs, circle graphs, and line graphs.

Mental Mathematics

11. Memorize or mentally calculate math facts and improve mental mathematics.
    11.1   Recall from memory multiplication facts (up to 10 x 10) and the corresponding division facts.
    11.2   Mentally add and subtract whole numbers to 100.
    11.3   Mentally multiply and divide whole numbers by 1, 10, and 100.

Mathematical Practices

12. Students should:
    12.1   *Solve word problems using mathematical concepts including*: addition, subtraction, multiplication, division, fractions, perimeter, area, length, capacity, weight, time, and money.
    12.2   *Use appropriate physical and conceptual tools including*: geometric pattern blocks, base ten place value blocks, fraction strips and circles, rulers, measuring cups, scales, coins, bills, and clocks.
    12.3   *Understand and use appropriate mathematical language and terminology including*: sum, difference, factor, product, quotient, ten-thousand, round, numerator, denominator, equivalent, and inverse.

12.4 *Create representations of mathematical scenarios, problems, and processes including*: arrays, area diagrams, place value charts, bar models, fraction strips and circles, tables, circle graphs, bar graphs, and line graphs.

12.5 *Reason about mathematical relationships and give evidence for conclusions by*: distinguishing relevant information from irrelevant information, estimating values, predicting answers, showing the steps used to solve a problem, checking answers using inverse operations, and using multiple methods to solve a single problem.

12.6 *Communicate mathematical thinking by*: writing and evaluating mathematical sentences, generating a solution method and predicting whether it would work at other times and in other problem contexts, and explaining why an answer does or does not make sense.

History of Mathematics

13. Read or listen to true stories of famous mathematicians. Such as:

13.1 Zeno of Elea and the paradox of Achilles and the tortoise.

13.2 The Library of Alexandria, Hypatia of Alexandria, and the teaching of mathematics.

13.3 Roman numerals and the non-positional Roman system of numbering.

13.4 The Salamis tablet and the development of the abacus.

14. Explore the nature and purpose of mathematics.

14.1 Describe the differences between arithmetic and geometry and discuss why each is an essential part of mathematics.

14.2 Connect mathematics to art by investigating various symmetries in painting and sculpture.

14.3 Imagine the nature of infinity (∞) by trying to find the largest number there is.

Grade Four

Numbers

1. Understand and use whole numbers to 1,000,000.
    1.1   Count, read, and write whole numbers to 1,000,000 using numerals and words.
    1.2   Write whole numbers to 1,000,000 in expanded form and write numerals for numbers expressed in expanded form.
    1.3   Identify the place value of each digit for whole numbers to 1,000,000.
    1.4   Compare whole numbers to 1,000,000 using the symbols <, >, and =.
    1.5   Arrange a group of whole numbers to 1,000,000 in increasing or decreasing order.
    1.6   Place whole numbers to 1,000,000 on a number line.
    1.7   Round whole numbers to 1,000,000 to any place value.
    1.8   Find all factor pairs for whole numbers to 100.
    1.9   Find multiples of single-digit whole numbers.
    1.10 Understand a *prime* number to be a number that is only divisible by 1 and itself.
    1.11 Understand a *composite* number to be a number that is divisible by a number other than 1 and the number itself.
    1.12 Identify whole numbers to 100 as prime or composite.

2. Identify and describe fractions and decimals.
    2.1   Name and draw representations of fractions, including fractions greater than one (improper fractions) and mixed numbers.
    2.2   Place fractions on a number line.
    2.3   Compare fractions to the benchmark fractions 0, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and 1.
    2.4   Compare fractions with the same and different denominators (like and unlike fractions) using the symbols <, >, and =.
    2.5   Break apart fractions into sums of fractions with the same denominators and smaller numerators.
    2.6   Convert improper fractions into mixed numbers and mixed numbers into improper fractions.
    2.7   Find equivalent fractions and reduce fractions to their simplest form.

2.8   Understand a *decimal* number to be a number with a whole part and a fractional part separated by a decimal point (.).

2.9   Read, write, and describe the value of decimal numbers to two decimal places.

2.10   Place decimal numbers to two decimal places on a number line.

2.11   Compare decimal numbers to two decimal places using the symbols <, >, and =.

2.12   Break apart and put together decimal numbers into various combinations of wholes, tenths, and hundredths.

2.13   Round decimal numbers to the nearest tenth and the nearest whole number.

2.14   Know that fractions and decimals each describe parts of a whole.

2.15   Distinguish between situations when it is more appropriate to use fractions and situations when it is more appropriate to use decimals (e.g., $\frac{1}{3}$ of a sandwich instead of $.\overline{333}$, and \$4.25 instead of $\$4\frac{1}{4}$).

2.16   Convert between decimal and fractional notation (with denominators of 2, 4, 5, 10, and 100).

2.17   Compare fractions, mixed numbers, and decimals to two decimal places using the symbols <, >, and =.

Arithmetic

3.   Add and subtract whole numbers to 1,000,000.

    3.1   Add whole numbers to 1,000,000 using the standard algorithm.

    3.2   Subtract whole numbers to 1,000,000 using the standard algorithm.

4.   Multiply and divide whole numbers within 1,000,000.

    4.1   Multiply whole numbers within 1,000,000 (multi-digit numbers by one- and two-digit numbers) using the standard algorithm.

    4.2   Divide whole numbers within 1,000,000 (multi-digit numbers by one-digit numbers) using the standard algorithm.

    4.3   Understand a *remainder* as a leftover of the dividend, and know that it can also be written as a fraction with the divisor as the denominator.

    4.4   Solve problems using the distributive property $[a * (b + c) = a * b + a * c]$.

5.   Add, subtract, and multiply fractions.

    5.1   Add and subtract fractions with different denominators (unlike fractions) by making common denominators.

    5.2   Add fractions to whole numbers and subtract fractions from whole numbers.

    5.3   Add and subtract mixed numbers and improper fractions.

    5.4   Multiply a fraction by a whole number and a whole number by a fraction.

6. Add, subtract, multiply, and divide decimals to two decimal places.
    6.1   Add and subtract decimals using the standard algorithm.
    6.2   Multiply decimals by whole numbers using the standard algorithm.
    6.3   Divide decimals by whole numbers using the standard algorithm.

Geometry

7. Identify, describe, compare, and draw common geometrical objects.
    7.1   Understand a *circle* to be the set of all the points equidistant from a fixed point called the center.
    7.2   Identify the radius, diameter, and circumference of a circle.
    7.3   Draw circles with a compass.
    7.4   Identify, describe, and draw right, acute, obtuse, straight, and reflex angles using a straightedge and a protractor.
    7.5   Measure angles in degrees using a protractor.
    7.6   Add and subtract angle measures.
    7.7   Identify, describe, and draw equilateral, isosceles, scalene, right, acute, and obtuse triangles.

8. Find the area and perimeter of shapes.
    8.1   Find the perimeter of any two-dimensional shape.
    8.2   Find an unknown side length of a two-dimensional shape given the perimeter and the other sides.
    8.3   Find an unknown side length of a rectangle given the area and other sides.
    8.4   Find the area of composite figures made up of rectangles and squares.

Measurement

9. Measure and compare objects.
    9.1   Measure length, weight, mass, and capacity using fractional and decimal values.

10. Represent and analyze data.
    10.1   Collect a set of numerical data using a simple survey, experiment, or other means.
    10.2   Select appropriate graphs or charts (table, bar graph, circle graph, and/or line graphs) to represent a given set of data.
    10.3   Make a prediction using a graph or chart, identifying whether an event is certain, likely (probable), 50/50, unlikely (improbable), or impossible.

Mental Mathematics

11. Memorize or mentally calculate math facts and improve mental mathematics.
     11.1   Recall from memory all single-digit addition, subtraction, multiplication, and division facts.
     11.2   Mentally add and subtract two whole numbers to 1,000.
     11.3   Mentally multiply and divide whole numbers by any power of 10.

Mathematical Practices

12. Students should:
     12.1   *Solve word problems using mathematical concepts including*: addition, subtraction, multiplication, division, fractions, decimals, perimeter, area, length, capacity, and weight.
     12.2   *Use appropriate physical and conceptual tools including*: fraction strips and circles, rulers, measuring cups, scales, compasses, straightedges, and protractors.
     12.3   *Understand and use appropriate mathematical language and terminology including*: factor, multiple, prime, composite, hundred-thousand, million, decimal place, tenth, hundredth, and remainder.
     12.4   *Create representations of mathematical scenarios, problems, and processes including*: arrays, area diagrams, place value charts, bar models, fraction strips and circles, tables, charts, bar graphs, circle graphs, and line graphs.
     12.5   *Reason about mathematical relationships and give evidence for conclusions by*: using letters, boxes, or other symbols to represent a missing number in simple mathematical sentences, predicting answers, showing work on multi-step problems, checking answers using inverse operations, using multiple methods to solve a single problem, and breaking complex problems into simpler problems.
     12.6   *Communicate mathematical thinking by*: writing and evaluating mathematical sentences, discussing mathematical relationships and patterns, and explaining why an answer does or does not make sense.

History of Mathematics

13. Read or listen to true stories of famous mathematicians. Such as:
     13.1   Babylonian mathematics and the division of the circle into 360°
     13.2   Eratosthenes and the sieve method for finding prime numbers
     13.3   Euclid and the use of logic and deductive reasoning to create a detailed system of geometry in his *Elements*
     13.4   Leonhard Euler and the puzzle of the 7 Bridges of Königsberg

14. Explore the nature and purpose of mathematics.
    14.1   Describe the ways in which mathematics is an art and the ways in which mathematics is a science.
    14.2   Connect mathematics to literature by considering the ways mathematics makes use of literary metaphors and the ways literature makes use of mathematical metaphors.
    14.3   Imagine whether mathematical truths would still exist if there were no people to think of them/discover them.

Grade Five

Numbers

1. Understand and use whole numbers.
   1.1 Know that the place value system continues indefinitely with each additional place to the left representing 10 times the value of the previous place.
   1.2 Read, write, and describe the value of whole numbers greater than one million, including numbers in the billions and trillions.
   1.3 Find the prime factorization of a whole number.
   1.4 Write a whole number as a product of prime factors using exponents to show factor multiples.
   1.5 Find the greatest common factor (GCF) of two or more whole numbers.
   1.6 Find the least common multiple (LCM) of two or more whole numbers.

2. Identify and describe fractions, decimals, and percentages.
   2.1 Read, write, and describe the value of decimal numbers to three decimal places.
   2.2 Order and compare decimal numbers to three decimal places.
   2.3 Understand *percentage* as part of a hundred and represent a percentage with the percent sign (%).
   2.4 Express fractions and decimals as percentages.
   2.5 Express percentages as fractions and as decimals.

3. Understand and use negative numbers.
   3.1 Understand a *negative* number to be a number less than 0.
   3.2 Understand an *integer* to be a number with no decimal or fractional part, and know that the set of integers includes positive whole numbers, negative whole numbers, and 0.
   3.3 Arrange a group of integers, including negative numbers, in increasing or decreasing order.
   3.4 Place integers, including negative numbers, on a number line.
   3.5 Compare integers, including negative numbers, using the symbols <, >, and =.

4. Identify and describe exponents.
    4.1   Understand *exponents* as notation describing repeated multiplication of the same number.
    4.2   Identify the base and the exponent (power) of an exponential expression.
    4.3   Know that *square* refers to a power of 2 and that *cube* refers to a power of 3.
    4.4   Write positive integer powers of whole numbers in standard form.

5. Identify and describe ratios and rates.
    5.1   Understand a *ratio* as a comparison of two quantities.
    5.2   Represent ratios in multiple forms (e.g., 2:3, 2 to 3, $\frac{2}{3}$).
    5.3   Distinguish between part-part ratios and part-whole ratios.
    5.4   Write a ratio to describe a relationship between two quantities.
    5.5   Understand a *rate* as a ratio of two quantities with different units (e.g., 2 eggs for every 3 cups of flour).
    5.6   Understand a *unit rate* as a ratio with a denominator of 1 (e.g., miles per hour, price per item).
    5.7   Write rates and unit rates and determine the units of a unit rate.

Arithmetic

6. Multiply and divide whole numbers.
    6.1   Multiply whole numbers (multi-digit numbers by multi-digit numbers) using the standard algorithm.
    6.2   Divide whole numbers (multi-digit numbers by multi-digit numbers) using the standard algorithm.
    6.3   Understand and use the *distributive property* $[a * (b + c) = a * b + a * c]$.

7. Add, subtract, multiply, and divide fractions.
    7.1   Add and subtract like fractions, unlike fractions, mixed numbers, and improper fractions.
    7.2   Multiply fractions by whole numbers, mixed numbers, and other fractions.
    7.3   Divide fractions by whole numbers.

8. Add, subtract, multiply, and divide decimals to three decimal places.
    8.1   Add and subtract decimals using the standard algorithm.
    8.2   Multiply decimals by whole numbers and by other decimals using the standard algorithm.
    8.3   Divide decimals by whole numbers and by other decimals using the standard algorithm.

Geometry

9. Identify, describe, compare, and draw common geometrical objects.
    - 9.1   Know that the angles of any triangle add up to 180°.
    - 9.2   Know that the angles of any quadrilateral add up to 360°.
    - 9.3   Find the measure of an unknown angle of a triangle or quadrilateral given the measures of the other angles.

10. Find the perimeter, area, and volume of shapes.
    - 10.1   Find the area of rectangles, triangles, and parallelograms
    - 10.2   Find the unknown dimension of a rectangle, triangle, or parallelogram given the area and one dimension.
    - 10.3   Understand *volume* to be the space inside a three-dimensional object.
    - 10.4   Find the volume of cubes and rectangular prisms.
    - 10.5   Find the volume of composite shapes composed of cubes and rectangular prisms.
    - 10.6   Find the unknown dimension of a rectangular prism given the volume and the other two dimensions.
    - 10.7   Select and use appropriate units of measure (e.g., inches, square inches, cubic inches) for perimeter, area, and volume.

11. Understand and use the x-y coordinate plane.
    - 11.1   Understand the *coordinate plane* to be a diagram for graphing two related pieces of information with two perpendicular number lines.
    - 11.2   Identify the origin, x-axis, and y-axis on the coordinate plane.
    - 11.3   Identify x- and y-coordinates of points on a coordinate plane.
    - 11.4   Identify and name the four quadrants of the coordinate plane.
    - 11.5   Graph ordered pairs in all four quadrants of the coordinate plane.

Measurement

12. Measure and compare objects.
    - 12.1   Convert length, weight, mass, capacity, and time measurements from larger to smaller units (e.g., feet to inches) and vice versa within English and metric systems.

13. Represent and analyze data.
    - 13.1   Collect a set of numerical data using a simple survey, experiment, or other means.
    - 13.2   Find the mean, median, mode, and range of a set of numerical data.

Mental Mathematics

14. Memorize or mentally calculate math facts and improve mental mathematics.
    - 14.1 Recall from memory common fraction to decimal conversions.
    - 14.2 Recall from memory common measurement conversions within the English and metric systems.
    - 14.3 Mentally estimate addition, subtraction, multiplication, and division problems with large numbers.

Mathematical Practices

15. Students should:
    - 15.1 *Solve word problems using mathematical concepts including*: fractions, decimals, percents, ratios, rates, perimeter, area, volume, surface area, and angles.
    - 15.2 *Use appropriate physical and conceptual tools including*: protractors, the coordinate plane, graphs, and charts.
    - 15.3 *Understand and use appropriate mathematical language and terminology including*: prime factorization, greatest common factor, least common multiple, million, billion, trillion, thousandth, exponent, power, base, coordinate plane, axis, origin, x-coordinate, y-coordinate, quadrant, mean, median, mode, and range.
    - 15.4 *Create representations of mathematical scenarios, problems, and processes including*: place value charts, bar models, fraction strips and circles, diagrams, and formulas.
    - 15.5 *Reason about mathematical relationships and give evidence for conclusions by*: using variables in simple mathematical sentences, showing work on multi-step problems, checking answers using inverse operations, and applying strategies derived from simple problems to solve complex problems.
    - 15.6 *Communicate mathematical thinking by*: writing and evaluating mathematical sentences that use parentheses to indicate which operation to perform first, discussing mathematical relationships and patterns, and explaining why an answer does or does not make sense.

History of Mathematics

16. Read or listen to true stories of famous mathematicians. Such as:
    - 16.1 Pythagoras and harmonic ratios in music.
    - 16.2 Luca Pacioli and the golden ratio.
    - 16.3 Brahmagupta and the arithmetic of negative numbers.
    - 16.4 René Descartes and the invention of the coordinate plane.

17. Explore the nature and purpose of mathematics.

    17.1   Describe the reasons that negative numbers might be useful in mathematics and where we see negative numbers in the world around us.

    17.2   Connect mathematics to music by investigating scales, intervals, and frequency ratios.

    17.3   Imagine what a world without mathematics might look like and what consequences would occur.

Grade Six

Numbers and Operations

1. Understand and use rational numbers.
    1.1 Understand a *rational number* to be a number that can be expressed as a fraction of two integers.
    1.2 Know that the set of rational numbers includes positive integers, negative integers, zero, fractions, mixed numbers, terminating decimals, and repeating decimals.
    1.3 Compare rational numbers using the symbols <, >, and =.
    1.4 Arrange a group of rational numbers in increasing or decreasing order.
    1.5 Place rational numbers on a number line.
    1.6 Understand the *absolute value* of a rational number to be its distance from zero on a number line.
    1.7 Determine the absolute value of a rational number by finding its distance from zero.

2. Identify and describe fractions, decimals, and percentages.
    2.1 Understand the *reciprocal* of a number (other than 0) to be 1 over that number.
    2.2 Know that the product of a number and its reciprocal is equal to 1.
    2.3 Determine the reciprocal of a fraction.

3. Identify and describe exponents and roots.
    3.1 Write positive integer powers of whole numbers in standard form.
    3.2 Understand the *square root* of a number to be the number that when multiplied by itself produces the original number.
    3.3 Find the square root of perfect squares within 400.

4. Identify and describe ratios and rates.
    4.1 Solve for unknown quantities using ratios, rates, and unit rates.
    4.2 Know and use common rates including speed, price, and salary.

5. Add, subtract, multiply, and divide fractions.
    5.1   Add and subtract like fractions, unlike fractions, mixed numbers, and improper fractions.
    5.2   Multiply fractions by whole numbers, mixed numbers, and other fractions.
    5.3   Know that dividing by a fraction is equivalent to multiplying by its reciprocal.
    5.4   Divide fractions by whole numbers and whole numbers by fractions.
    5.5   Divide fractions by other fractions.

6. Add, subtract, multiply, and divide decimals.
    6.1   Add and subtract decimals to three decimal places.
    6.2   Multiply and divide decimals by whole numbers and other decimals to three decimal places.

7. Add, subtract, multiply, and divide positive and negative numbers.
    7.1   Add and subtract positive and negative integers.
    7.2   Multiply and divide positive and negative integers.

Pre-Algebra

8. Understand variables and formulas.
    8.1   Understand a *variable* to be a letter or other symbol standing for a changeable or unknown quantity.
    8.2   Recognize unknown quantities in a problem and assign appropriate variables.
    8.3   Understand a *formula* to be a set of symbols that expresses a mathematical relationship, law, or principle.
    8.4   Write a mathematical formula for a given set of conditions.

9. Simplify and evaluate expressions.
    9.1   Understand an *expression* to be a mathematical phrase that contains numbers, variables, operation signs, and/or grouping symbols such as parentheses but does not contain an equal sign.
    9.2   Write expressions with and without a variable.
    9.3   Know that parentheses are grouping symbols that indicate which operations should be performed first, and recognize different kinds of parentheses (brackets, braces, and/or a vinculum).
    9.4   Know that natural precedence determines the order of operations when there are no grouping symbols.
    9.5   Simplify expressions by using the order of operations (parentheses, exponents, multiplication/division, addition/subtraction).

9.6   Simplify expressions by applying properties of rational numbers (commutative, associative, distributive, identity, inverse).

9.7   Evaluate an expression with a variable by substituting a numerical value for the variable.

10. Solve equations.

10.1   Understand an *equation* to be a mathematical sentence that relates two equal expressions with an equal sign.

10.2   Distinguish between an expression and an equation.

10.3   Determine if a potential value for the variable in an equation makes the equation true or false.

10.4   Write one-step equations with one variable to represent a written or pictorial scenario.

10.5   Solve one-step equations with one variable using inverse operations.

Geometry

11. Identify, describe, compare, and draw common geometrical objects.

11.1   Identify, describe, and draw elements of circles, including center, radius, diameter, and chord.

11.2   Identify, describe, and draw vertical, adjacent, complementary and supplementary angles.

11.3   Find unknown angle values in a pair of complementary or supplementary angles.

11.4   Describe two-dimensional shapes by their edges and vertices, and describe three-dimensional shapes by their faces, edges, and vertices.

11.5   Understand a *net* to be a two-dimensional representation that can be folded up to form a three-dimensional shape.

11.6   Identify or form a three-dimensional shape represented by a given net.

11.7   Identify, draw, and describe nets of various three-dimensional shapes.

11.8   Sketch a geometric shape given a set of conditions (e.g., draw a triangular prism with a right triangular base).

12. Find the perimeter, area, and surface area of shapes.

12.1   Understand π (pi) to be the constant ratio of a circle's circumference to its diameter.

12.2   Know and use numerical approximations for π, including 3.14 and 22/7.

12.3   Derive and use formulas for the circumference ($C = 2\pi r$) and area ($A = \pi r^2$) of a circle, where r is the circle's radius.

12.4   Understand *surface area* to be the combined area of the faces on the outside of a three-dimensional shape.

12.5   Find the surface area of cubes and rectangular prisms.

12.6   Find the surface area of composite shapes composed of rectangular prisms.

13. Understand and use the x-y coordinate plane.
    13.1   Graph ordered pairs in all four quadrants of the coordinate plane.
    13.2   Reflect an ordered pair across the x-axis or y-axis.
    13.3   Draw polygons in the coordinate plane given coordinates for the vertices.
    13.4   Reflect polygons across the x-axis or y-axis.

Statistics and Probability

14. Measure and compare objects.
    14.1   Measure quantities that can have opposite directions or values with positive and negative numbers, including temperature, elevation, credits/debits, and electric charge.
    14.2   Convert temperatures between Celsius and Fahrenheit degrees.

15. Analyze data using statistical measures.
    15.1   Understand *statistics* to be a branch of mathematics that involves the collection, representation, and analysis of quantitative data.
    15.2   Write statistical questions that require the collection of data.
    15.3   Collect data to answer statistical questions.
    15.4   Graph a set of data using a dot plot, histogram, and bar graph.
    15.5   Find and interpret the mean, median, mode, and range of a set of data.
    15.6   Distinguish between a measure of center (mean, median, and mode) and a measure of variation (range).

Mental Mathematics

16. Memorize or mentally calculate math facts to improve mental mathematics.
    16.1   Recall from memory formulas for the perimeters, areas, and volumes of shapes, including triangles, squares, rectangles, circles, cylinders, and rectangular prisms.
    16.2   Recall from memory or mentally calculate the squares of numbers from 0 to 20.

Mathematical Practices

17. Students should:
    17.1   *Solve word problems using mathematical concepts including*: fractions, decimals, percents, ratios, rates, exponents, roots, negative numbers, one-step equations, perimeter, area, volume, surface area, and angles.

17.2   *Use appropriate physical and conceptual tools including*: compass, protractor, the coordinate plane, graphs, and charts.

17.3   *Understand and use appropriate mathematical language and terminology including*: rational number, absolute value, expression, equation, inequality, reciprocal, variable, formula, term, net, statistics, data, and center.

17.4   *Create representations of mathematical scenarios, problems, and processes including*: expressions, equations, formulas, dot plots, histograms, bar graphs, bar models, and diagrams.

17.5   *Reason about mathematical relationships and give evidence for conclusions by*: manipulating equations by performing identical operations to both sides, showing work on multi-step problems, checking answers using inverse operations, using and interpreting a formula to answer questions (e.g., area = length x width), and investigating the relationship between two related quantities.

17.6   *Communicate mathematical thinking by*: writing and evaluating mathematical sentences using variables and the order of operations, translating written or verbal descriptions into algebraic expressions (and vice versa), using information from a graph to make a claim based on statistical data, evaluating the validity of a claim based on statistical data, and explaining why an answer does or does not make sense.

History of Mathematics

18. Read and discuss true stories of famous mathematicians. Such as:
   18.1   Archimedes and the approximation of using inscribed and circumscribed polygons.
   18.2   Diophantus and the solutions to equations with exponents.
   18.3   Democritus and the volumes of prisms and cones.
   18.4   John Arbuthnot, Nettie Stevens, and the discovery of the sex ratio.

19. Explore the nature and purpose of mathematics.
   19.1   Describe the nature of a formula and discuss why mathematics might lend itself so naturally to formulas.
   19.2   Connect mathematics to sports by investigating scoring, statistical analysis, and projectile motion.
   19.3   Imagine what would happen if there were no order of operations.

Grade Seven

Numbers and Operations

1. Understand and use rational and irrational numbers.
   1.1 Know that a rational number can be expressed as a terminating or repeating decimal.
   1.2 Understand an *irrational number* to be a non-terminating, non-repeating decimal that cannot be expressed as a fraction of two integers.
   1.3 Distinguish between rational and irrational numbers.
   1.4 Estimate the value of an irrational number.
   1.5 Arrange a group of irrational numbers in increasing or decreasing order.
   1.6 Place irrational numbers on a number line.
   1.7 Compare irrational numbers using the symbols <, >, and =.

2. Identify and describe exponents and roots.
   2.1 Understand a *root index* to be the number placed above the radical symbol indicating which root to find, and know that if the root index is not specified, it is assumed to be 2.
   2.2 Understand the *cube root* of a number to be the number that when cubed produces the original number.
   2.3 Find the cube root of perfect cubes within 400.
   2.4 Recognize the inverse relationship between raising a number to a power and taking a root of that number.
   2.5 Find approximations of the square roots and cube roots of integers within 400 that are not perfect squares or cubes.

3. Identify and describe ratios, rates, proportions, and percentages.
   3.1 Understand a *proportion* to be an equation of two ratios.
   3.2 Write proportions and solve for unknown quantities.
   3.3 Determine whether two quantities have a proportional relationship by examining a table of values, graph, or written description.
   3.4 Solve problems with proportions including scaling distances and similar shapes.
   3.5 Understand a *percentage* to be a ratio expressed as a fraction out of 100.
   3.6 Write percent equations and solve for unknown quantities.
   3.7 Calculate the percent increase and decrease of a quantity.

Pre-Algebra

4. Simplify and evaluate expressions.
    4.1   Understand an *expression* to be a mathematical phrase that contains numbers, variables, operation signs, and/or grouping symbols such as parentheses but does not contain an equal sign.
    4.2   Write expressions with and without variables.
    4.3   Simplify expressions, including those with positive and negative rational numbers, by using the order of operations.
    4.4   Simplify expressions by applying the properties of numbers, including commutative, associative, distributive, identity, and inverse.
    4.5   Distinguish between like terms and unlike terms.
    4.6   Simplify expressions by combining like terms.
    4.7   Evaluate algebraic expressions by substituting numerical values for the variable(s).

5. Solve one-variable equations and inequalities.
    5.1   Understand an *equation* to be a mathematical sentence that relates two equal expressions with an equal sign.
    5.2   Understand an *inequality* to be a mathematical sentence that relates two unequal expressions with the signs >, <, ≥, or ≤.
    5.3   Distinguish between an expression, an equation, and an inequality.
    5.4   Write algebraic equations with one variable.
    5.5   Solve one-step and two-step equations with one variable using inverse operations.
    5.6   Write algebraic inequalities with one variable.
    5.7   Solve simple one-step and two-step inequalities with one variable using inverse operations.
    5.8   Represent an inequality on a number line.
    5.9   Determine if a potential value for the variable in an equation or inequality makes the equation or inequality true or false.

6. Understand and represent two-variable equations.
    6.1   Understand an *equation with two variables* to be a statement expressing a relationship between two unknown quantities.
    6.2   Know that a solution to a two-variable equation is an ordered pair of values for the variables that makes the equation true.
    6.3   Distinguish between ordered pairs that satisfy the condition expressed by a two-variable equation and those that do not.
    6.4   Given an equation with two variables, generate a table of ordered pairs that satisfy a two-variable equation.

6.5   Graph ordered pairs that satisfy a two-variable equation on a coordinate plane and draw the line or curve that they determine.

6.6   Graph two equations on a coordinate plane and estimate their point of intersection.

Geometry

7.   Identify, describe, compare, and draw common geometrical objects.

7.1   Identify, describe, and draw equilateral, isosceles, scalene, right, acute, and obtuse triangles.

7.2   Identify, describe, and draw elements of triangles including base, height (altitude), leg, and hypotenuse.

7.3   Know and use the Pythagorean theorem (in a right-angled triangle, $a^2 + b^2 = c^2$) to find unknown side lengths in right triangles.

7.4   Show that a triangle is a right triangle by using the converse of the Pythagorean theorem.

7.5   Understand *congruent* figures to be geometric objects that have exactly the same size and shape.

7.6   Understand *similar* figures to be geometric objects that have the same shape but different sizes.

7.7   Know the properties of similar figures and solve for unknown side lengths using a proportion.

8.   Find the perimeter, area, volume, and surface area of shapes.

8.1   Know and use formulas for the perimeters and areas of triangles, squares, rectangles, trapezoids, and parallelograms.

8.2   Find the perimeter and area of composite two-dimensional shapes by breaking them into more elementary shapes, including triangles, squares, and rectangles.

8.3   Know and use formulas for the volumes and surface areas of prisms, cylinders, cubes, and spheres.

8.4   Find the volume and surface area of composite shapes by breaking them into prisms, cylinders, cubes, and spheres.

8.5   Select and use appropriate units of measure for perimeter, area, surface area, and volume, such as inches, square inches, and cubic inches.

8.6   Convert length, weight, area, and capacity/volume measurements from English units to metric units and from metric units to English units.

9.   Understand and use the x-y coordinate plane.

9.1   Graph ordered pairs in all four quadrants of the coordinate plane.

9.2   Find the horizontal distance and vertical distance between two points on the coordinate plane.

9.3   Find the distance between any two points on the coordinate plane using the Pythagorean Theorem.

Statistics and Probability

10. Analyze data using statistical measures.

    10.1   Understand *sampling* to be the selection of a smaller group (a set of individual cases) from a larger group (population).

    10.2   Identify different ways of sampling, including convenience sampling and random sampling, and know ways to make a sample more representative.

    10.3   Draw conclusions about a larger population using data from a representative sample.

    10.4   Find and interpret the mean, median, mode, and range of a set of data.

    10.5   Represent a set of data using a box plot, labeling quartiles and identifying outliers.

11. Understand probability.

    11.1   Understand *probability* as the likelihood or chance of an event occurring.

    11.2   Understand a *sample space* to be the set of all possible outcomes for a situation or experiment.

    11.3   List the outcomes in the sample space for a given situation or experiment.

    11.4   Calculate the probability for a single event by dividing the number of favorable outcomes by the number of total outcomes.

    11.5   Represent probabilities as ratios/fractions, as decimals between 0 and 1, and as percentages between 0 and 100.

Mathematical Practices

12. Students should:

    12.1   *Solve word problems using mathematical concepts including*: ratios, rates, proportions, percents, linear equations, right triangles, perimeter, area, volume, surface area, and probability. Solve problems involving ratios and rates, including speed, price, and salary. Solve problems that involve percentages including discounts, markups, commissions, tax, and interest.

    12.2   *Use appropriate physical and conceptual tools including:* the coordinate plane, graphs, and charts.

    12.3   *Understand and use appropriate mathematical language and terminology including*: real number, irrational number, proportion, function, like terms, unlike terms, similar, intersection, sampling, representative, set, probability, event, and outcome.

    12.4   *Create representations of mathematical scenarios, problems, and processes including*: graphs of equations on a coordinate plane, formulas, bar/tape models, scale drawings, pictures of shapes and solids, box plots, and diagrams.

    12.5   *Reason about mathematical relationships and give evidence for conclusions by*: manipulating equations by performing identical operations to both sides, showing work on

multi-step problems, checking answers using inverse operations, using and interpreting a formula to answer questions (e.g., area = length x width), relating the solutions to an algebraic equation with the graphical representation of those solutions on a coordinate plane, relating problems to one another by noticing similarities and differences in contexts and methods of solving them, and generalizing a particular solution to develop a formula.

12.6   *Communicate mathematical thinking by*: using estimation and mental mathematics to make a prediction and then comparing a solution to the original prediction, translating written or verbal descriptions into algebraic expressions (and vice versa), relating solutions back to the context of the problem, discussing the relative merits of different solution methods, and explaining why an answer does or does not make sense.

History of Mathematics

13. Read and discuss true stories of famous mathematicians. Such as:
    13.1   Thales of Miletus and the origins of mathematical problem-solving.
    13.2   Pythagoras, Euclid, Bhaskara, and proofs of the Pythagorean theorem.
    13.3   Aryabhata and the approximation of $\sqrt{2}$.
    13.4   Blaise Pascal and the origins of the field of probability.

14. Explore the nature and purpose of mathematics.
    14.1   Describe the ways in which mathematics can provide compelling answers to questions and the ways in which mathematics can be limited in the answers it provides.
    14.2   Connect mathematics to history by investigating how mathematics has evolved over time through various cultures.
    14.3   Imagine the future of mathematics and what might be next in the progression of mathematical history.

Grade Eight

Numbers and Operations

1. Understand and use real numbers.
    1.1 Understand a *real number* to be any number that can be represented on a number line.
    1.2 Know that the set of real numbers is made up of rational and irrational numbers.
    1.3 Describe a given number as real, rational, irrational, positive, negative, prime, composite, odd, even, an integer, a whole number, a digit, and/or a natural (counting) number.
    1.4 Recognize special relationships between numbers including factor, multiple, additive inverse, multiplicative inverse (reciprocal), square, square root, cube, and cube root.
    1.5 Generate a number that satisfies a given set of conditions (e.g., find an even, prime integer).

2. Identify and describe exponents, powers, and roots.
    2.1 Understand *negative exponents* as a way to represent the reciprocal of a base to a positive power.
    2.2 Understand *rational exponents* as a way to represent powers and roots together.
    2.3 Simplify numerical expressions with exponents, powers, and roots, including expressions with negative and rational exponents.
    2.4 Convert numbers into and out of scientific notation.

Pre-Algebra

3. Simplify and evaluate expressions.
    3.1 Write expressions with real numbers using zero, one, or more than one variable.
    3.2 Simplify expressions with real numbers by using the order of operations (parentheses, exponents, multiplication/division, addition/subtraction) and by applying properties of real numbers (commutative, associative, distributive, identity, inverse, etc.).
    3.3 Evaluate algebraic expressions by substituting a set of numerical values for the variable(s).

4. Understand and use polynomials.
    4.1   Understand a *polynomial* to be a sum of algebraic terms having variables, coefficients, exponents, and/or constants.
    4.2   Understand a *monomial* to be a polynomial with only one term.
    4.3   Add and subtract polynomials.
    4.4   Multiply polynomials by monomials.
    4.5   Multiply two binomials and square a binomial.
    4.6   Understand the factoring of a polynomial to be a reversal of the distributive property.
    4.7   Factor out the greatest common monomial factor of a polynomial.
    4.8   Factor quadratic trinomials of the form $x^2 + bx + c$ into two binomial factors of the form $(x + m)(x + n)$ where *m* and *n* are integers.

5. Solve equations and inequalities.
    5.1   Solve multi-step equations and inequalities with one variable using inverse operations.
    5.2   Understand a *compound (combined) inequality* to be a statement that joins two or more inequalities using the words "and" or "or".
    5.3   Solve compound inequalities and represent the solutions graphically.
    5.4   Understand a *quadratic equation* to be an equation where the highest exponent of the variable is 2.
    5.5   Solve simple quadratic equations of the form $x^2 + bx + c = 0$ by factoring.

6. Understand and represent linear equations.
    6.1   Graph ordered pairs that fit a linear equation in two variables on a coordinate plane and draw the line that they determine.
    6.2   Understand *slope* to be the constant rate of change for a line that is calculated by the vertical change of a line (rise) divided by its horizontal change (run).
    6.3   Find the slope between two points using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.
    6.4   Find the slope of a line from its graph, table, or equation.
    6.5   Draw a graph of a line given the slope and a point on the line, two points, an equation, or a table.
    6.6   Know that vertical lines have a slope that is undefined.
    6.7   Distinguish between lines whose slopes are positive, negative, zero, or undefined.
    6.8   Understand an *x-intercept* to be the point(s) where the graph of an equation crosses the x-axis and a *y-intercept* to be the point(s) where the graph of an equation crosses the y-axis.
    6.9   Find the x-intercept and y-intercept of a line from its table, graph, or equation.
    6.10   Know that the slopes of parallel lines are equal.
    6.11   Know that the slopes of perpendicular lines are reciprocals with opposite signs (the slopes have a product of -1).

7.   Solve systems of linear equations.
   7.1   Understand a *system of equations* to be a set of two or more equations to be solved together containing the same variables.
   7.2   Know that a solution to a system of two equations is an ordered pair that makes both the equations true, and know that a system of two equations can have one solution, no solutions, or infinitely many solutions.
   7.3   Estimate the solution to a system of linear equations by graphing the equations on a coordinate plane.
   7.4   Solve a system of linear equations with integer coefficients by substitution.
   7.5   Solve a system of linear equations with integer coefficients by combination (elimination).

8.   Identify and describe functions.
   8.1   Understand a *function* to be an equation, rule, or law that assigns to each input (independent value) exactly one output (dependent value).
   8.2   Distinguish between an independent variable and a dependent variable.
   8.3   Understand the *domain* as the set of inputs accepted by the function and the *range* as the set of outputs produced by the function.
   8.4   Determine whether a relationship is a function given a table, graph, or equation.
   8.5   Understand a *linear function* to be a relationship between two variables with a constant rate of change whose graph is a straight line on the coordinate plane.
   8.6   Distinguish between linear and nonlinear functions.

Geometry

9.   Identify, describe, compare, and draw common geometrical objects.
   9.1   Know that the sum of the measures of the interior angles of a triangle is 180°.
   9.2   Show that the sum of the measures of the interior angles of a polygon with n sides is $(n-2)*180°$ by breaking polygons into triangles.
   9.3   Know that the measure of an interior angle of a regular polygon with n sides is $\frac{(n-2)*180°}{n}$.
   9.4   Find the measure of an unknown angle of a shape given the measures of the other angles.

10.   Understand and use rigid transformations.
   10.1   Understand a *rigid transformation* to be a change in a shape that generates a congruent shape by preserving distances between vertices.
   10.2   Distinguish between rigid and non-rigid transformations.

10.3 Identify, draw, and describe the three types of rigid transformations: *rotations, reflec-tions,* and *translations*.

10.4 Identify, draw, and describe mathematical *dilation* as a non-rigid transformation that generates a similar shape.

10.5 Describe a sequence of transformations that superimposes one similar shape onto another.

10.6 Depict rigid transformations on the coordinate plane by describing the effect of each type of transformation on the ordered pairs of the shape's vertices.

Probability and Statistics

11. Analyze data using statistical measures.

11.1 Understand *bivariate data* to be information about two potentially related variables (e.g., the relationship between ice cream sales and temperature).

11.2 Conduct an experiment to collect a set of bivariate data.

11.3 Create a scatter plot to represent a set of bivariate data.

11.4 Describe patterns in a set of bivariate data including clustering, outliers, positive and negative association, and linear and nonlinear association.

11.5 Roughly fit a straight line to a scatter plot with linear association.

12. Understand probability.

12.1 Understand a *set* to be a collection of distinct objects.

12.2 Know the terminology and notation of sets, including elements ($\in$), subsets ($\subset$), and the empty/null set ($\{\,\}, \varnothing$).

12.3 Understand *probability* as the likelihood or chance of an event occurring.

12.4 Know that if p is the probability of an event, then $1 - p$ is the probability of the event not occurring.

12.5 Run a simulation to test a probability model and compare the experimental results with the theoretical probabilities.

12.6 Understand a *permutation* to be an arrangement of a group of objects in a particular order.

12.7 Understand a *combination* to be a selection of a group of objects from a larger group without regard to order.

12.8 Distinguish between permutations and combinations.

12.9 Count the number of possible arrangements of a group of objects using permutations and combinations.

Mathematical Practices

13. Students should:

   13.1 *Solve word problems using mathematical concepts including*: proportions (direct and inverse), linear equations, quadratic equations, systems of equations, parallel lines, bivariate data, and probability.

   13.2 *Use appropriate physical and conceptual tools including*: protractor, the coordinate plane, variables, expressions, equations, tables, graphs, charts, diagrams, and formulas.

   13.3 *Understand and use appropriate mathematical language and terminology including*: directly proportional, inversely proportional, coefficient, constant, dependent variable, independent variable, domain, range, x-intercept, y-intercept, linear, slope, rise, run, quadratic, polynomial, monomial, binomial, trinomial, factoring, system of equations, transformation, rotation, reflection, translation, dilation, bivariate, permutation, and combination.

   13.4 *Create representations of mathematical scenarios, problems, and processes including*: graphs of equations on a coordinate plane, factoring boxes, function mapping diagrams, pictures of rigid transformations, scatter plots, and probability trees.

   13.5 *Reason about mathematical relationships and give evidence for conclusions by*: drawing diagrams to illustrate problem contexts, using inverse operations to "undo" a mathematical process, using and interpreting a formula to answer questions (e.g., area = length x width), relating the solutions to an algebraic equation with the graphical representation of those solutions on a coordinate plane, and making connections between related problems, methods, and concepts.

   13.6 *Communicate mathematical thinking by*: explaining the steps of a solution process, identifying key moments and potential pitfalls in a mathematical method, relating solutions back to the context of the problem, discussing the relative merits of different solution methods, and explaining why an answer does or does not make sense.

History of Mathematics

14. Read and discuss true stories of famous mathematicians. Such as:

   14.1 Pierre de Fermat, Sophie Germain, Andrew Wiles, and the history of Fermat's Last Theorem.

   14.2 Gottfried Leibniz and the development of the concept of a function.

   14.3 Gerolamo Cardano, gaming, and the birth of probability theory.

   14.4 Charles Babbage and Ada Lovelace, and the mathematical foundations of binary numbers and computing.

15. Explore the nature and purpose of mathematics.
    15.1   Describe the ways in which "thinking mathematically" affects our ability to make sense of societal and cultural affairs.
    15.2   Connect mathematics to social sciences by investigating the ways that bivariate relationships between different variables within a social context (e.g., income, education, age, and ethnicity) reveal trends and patterns within a population.
    15.3   Imagine numbers that are not "real" (i.e., neither rational nor irrational).

Algebra I

Language of Algebra

1. Axiomatic Systems
   1.1 Understand an *axiom* to be a fundamental truth or assumption that is accepted without proof.
   1.2 Know the axioms of real numbers including: reflexive, symmetric, transitive, commutative, associative, and distributive.
   1.3 Understand a *property* to be a statement that can be proven or shown to be true based on a set of axioms.
   1.4 Know the properties of real numbers including: addition property of equality, multiplication property of equality, multiplication property of zero, etc.
   1.5 Distinguish between an axiom and a property.
   1.6 **CHALLENGE TOPIC**: Justify the reasons for the steps in a mathematical argument or proof by appealing to an axiom, a property, or a previous step.

2. Numbers
   2.1 Describe a given number as real, rational, irrational, positive, negative, prime, composite, odd, even, an integer, a whole number, and/or a natural (counting) number.
   2.2 Recognize special relationships between two numbers including factor, multiple, additive inverse, multiplicative inverse (reciprocal), root, square, and cube.
   2.3 Given a set of conditions, generate a number that satisfies those conditions (e.g., find an even, prime integer).
   2.4 Understand the *absolute value* of a real number to be its distance from zero on a number line.
   2.5 Determine the absolute value of a real number.

3. Variables and Formulas
   3.1 Understand a *variable* to be a letter or other symbol standing for a changeable or unknown quantity.
   3.2 Understand a *constant* to be a fixed numerical value that never changes.
   3.3 Understand a *coefficient* to be a number that is multiplied by a variable.
   3.4 Understand a *formula* to be a set of symbols that expresses a mathematical relationship, law, or principle.

Expressions, Equations, and Inequalities

4. Expressions
   4.1 Understand an *expression* to be a mathematical phrase containing numbers, variables, operation signs, and/or grouping symbols such as parentheses, but which does not contain an equal sign.
   4.2 Write expressions using zero, one, or more than one variable.
   4.3 Simplify expressions with real numbers by using the order of operations and by applying properties of real numbers.
   4.4 Evaluate algebraic expressions by substituting a set of numerical values for the variable(s).

5. Equations and Inequalities
   5.1 Understand an *equation* to be a mathematical sentence that relates two equal expressions with the equal sign.
   5.2 Understand an *inequality* to be a mathematical sentence that relates two unequal expressions with the signs >, <, ≥, or ≤.
   5.3 Write algebraic equations and inequalities with one or more than one variable.
   5.4 Solve multi-step equations with one variable using inverse operations.
   5.5 Solve absolute value equations.
   5.6 Understand a *compound (combined) inequality* to be a statement that joins two or more inequalities using the words "and" or "or".
   5.7 Solve and graph compound inequalities.
   5.8 Solve and graph absolute value inequalities.
   5.9 Solve an equation with more than one variable for a particular variable.

6. Exponents, Powers, and Roots
   6.1 Derive and use the laws of exponents, including the product law ($a^m * a^n = a^{m+n}$), quotient law ($a^m \div a^n = a^{m-n}$), zero exponent law ($a^0 = 1$), negative exponent law ($a^{-m} = \frac{1}{a^m}$), power of a product law [$(ab)^n = a^n b^n$], power of a quotient law [$(a \div b)^n = a^n \div b^n$], and power of a power law [$(a^n)^m = a^{nm}$].
   6.2 Simplify numerical and algebraic expressions with exponents, powers, and roots, including expressions with negative and rational exponents.
   6.3 Know the square root of x^2 is the absolute value of x $\left(\sqrt{x^2} = |x| \right)$.
   6.4 Know that $x^2 = a$ has two solutions, $x = \sqrt{a}$ and $x = -\sqrt{a}$ and represent them as $x = \pm\sqrt{a}$.
   6.5 Convert numbers into and out of scientific notation.
   6.6 Multiply and divide numbers in scientific notation.

7. Polynomial Expressions

    7.1   Understand a *polynomial* to be a sum of algebraic terms having variables, coefficients, exponents, and/or constants.

    7.2   Classify polynomials by degree and number of terms.

    7.3   Add, subtract, and multiply polynomials.

    7.4   Divide polynomials by monomials.

    7.5   Raise polynomials to a power by expanding and combining like terms.

    7.6   Understand the *factoring* of a polynomial to be a reversal of the distributive property.

    7.7   Factor out the greatest common monomial factor of a polynomial.

    7.8   Factor general quadratic trinomials of the form $ax^2 + bx + c = 0$.

    7.9   Multiply conjugate binomials and factor a difference of squares.

    7.10 Square a binomial and factor a perfect square trinomial.

8. Quadratic Equations

    8.1   Understand a *quadratic equation* to be an equation where the highest exponent of the variable is 2.

    8.2   Know that the general form of a quadratic equation is $ax^2 + bx + c = 0$.

    8.3   Solve quadratic equations by factoring.

    8.4   Solve quadratic equations by completing the square.

    8.5   Derive the quadratic formula ($x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$).

    8.6   Solve quadratic equations using the quadratic formula.

9. Rational Expressions and Equations

    9.1   Understand a *rational expression* to be a quotient of polynomials.

    9.2   Simplify rational expressions by the rules for adding, subtracting, multiplying, and dividing fractions.

    9.3   Understand a *rational equation* to be an equation of rational expressions.

    9.4   Solve rational equations and check for extraneous solutions.

10. Radical Expressions and Equations

    10.1 Understand a *radical expression* to be an expression containing a radical sign, radicand, and root index.

    10.2 Derive and use the properties of radicals, including: the product property ($\sqrt[n]{a} * \sqrt[n]{b} = \sqrt[n]{ab}$), the quotient property ($\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$), and the power property ($\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$).

    10.3 Write radical expressions in simplest radical form.

    10.4 Add, subtract, multiply, and divide numeric and algebraic radical expressions with one or more terms.

    10.5 Rationalize a radical expression with a binomial in the denominator by multiplying the numerator and denominator by the conjugate of the denominator.

    10.6 Understand a *radical equation* to be an equation with a variable inside the radicand.

    10.7 Solve radical equations and check for extraneous solutions.

Functions

11. Functions

11.1   Understand a *function* to be a relationship or rule that assigns to each input (independent value) exactly one output (dependent value).

11.2   Determine whether a relationship is a function given a graph (using the vertical-line test), an equation, or a table of values.

11.3   Distinguish between an independent variable and a dependent variable.

11.4   Find the x-intercept(s) and y-intercept of a function from an equation, graph, or table.

11.5   Understand the *domain* as the set of inputs accepted by the function or relation and the *range* as the set of outputs produced by the function or relation.

11.6   Find the domain and range of a function.

11.7   Convert between different functional notations, including $y =$ and $f(x) =$.

11.8   Translate between various ways of representing functions: algebraic (equation/formula), visual (graph), numerical (table), and written/verbal (scenario).

11.9   Recognize and categorize different types of functions, including: linear, quadratic, and nonlinear/non-quadratic.

12. Linear Functions

12.1   Understand a *linear function* to be a relationship between two variables with a constant rate of change whose graph is a straight line on the coordinate plane.

12.2   Extend a linear pattern on a coordinate plane.

12.3   Understand *slope* to be the constant rate of change for a line that is calculated by the vertical change of a line (rise) divided by its horizontal change (run).

12.4   Find the slope of the line between two points using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

12.5   Find the slope of a linear function from its table, graph, or equation.

12.6   Write the equation of a linear function in slope-intercept form ($y = mx + b$), standard form ($Ax + By = C$), and point-slope form ($y - y_1 = m(x - x_1)$).

12.7   Know that a vertical line is not the graph of a function and has an undefined slope.

12.8   Solve and graph linear inequalities and systems of linear inequalities.

12.9   **CHALLENGE TOPIC**: Draw a scatter plot from a given data set, draw an estimated line of best fit (regression line), and find the equation of that line (regression equation).

13. Quadratic Functions

13.1   Understand a *quadratic function* to be a polynomial function in which the highest power of the independent variable is 2 and whose graph is called a parabola.

13.2   Graph quadratic functions by generating a table of values from the equation.

13.3   Graph quadratic functions using transformations, including horizontal and vertical translations, and vertical reflections.

13.4   Graph quadratic functions by finding the line of symmetry, vertex, y-intercept, and x-intercept(s) (if any).

14. Variation
    14.1   Distinguish between direct variation (direct proportionality) and inverse variation (inverse proportionality).
    14.2   Determine the constant of proportionality in a proportional relationship.
    14.3   Write functions for direct proportions and inverse proportions.
    14.4   Represent functions for direct proportions and inverse proportions on a graph.

15. Systems of Equations
    15.1   Understand a *system of equations* to be a set of two or more equations with the same variables.
    15.2   Know that a solution to a system of equations is a set of values for the variables that makes all the equations true.
    15.3   Solve linear systems of equations with two variables by graphing the equations on a coordinate plane and looking for the point of intersection.
    15.4   Solve linear systems of equations with two variables by substitution.
    15.5   Solve linear systems of equations with two variables by combination (elimination).

Mathematical Practices

16. Students should:
    16.1   *Solve word problems using mathematical concepts including*: proportions (direct and inverse), linear equations, quadratic equations, vertical motion problems, radical equations, the Pythagorean Theorem, systems of equations, mixture problems, bivariate data, and probability.
    16.2   *Use appropriate physical and conceptual tools including*: the coordinate plane, variables, expressions, equations, tables, graphs, charts, diagrams, formulas, and scientific calculators.
    16.3   *Understand and use appropriate mathematical language and terminology including*: directly proportional, inversely proportional, domain, range, x-intercept, y-intercept, quadratic, polynomial, monomial, factoring, system of equations, transformation, rotation, reflection, translation, dilation, bivariate, permutation, and combination.
    16.4   *Create representations of mathematical scenarios, problems, and processes including*: visual, symbolic, verbal, contextual, and physical representations of a given problem, graphs on a coordinate plane, inequality graphs, function mapping diagrams, and function tables.
    16.5   *Reason about mathematical relationships and give evidence for conclusions by*: drawing diagrams to illustrate problem contexts, using inverse operations to "undo" a mathematical process, using and interpreting a formula to answer questions (e.g., area = length x width), and making connections between related problems, methods, and concepts.

16.6 *Communicate mathematical thinking by*: explaining the steps of a solution process, identifying key moments and potential pitfalls in a mathematical method, relating solutions back to the context of the problem, discussing the relative merits of different solution methods, and explaining why an answer does or does not make sense.

History of Mathematics

17. Read and discuss true stories of famous mathematicians. Such as:
17.1 Al-Khwarizmi and the development of algebra.
17.2 Francois Viete and the use of variables in mathematics.
17.3 Leonhard Euler and the codification of algebra in his *Elements of Algebra*.
17.4 Blaise Pascal and binomial expansions with Pascal's triangle.

18. Explore the nature and purpose of mathematics.
18.1 Describe the importance of checking to make sure an answer makes sense within the original problem context.
18.2 Connect mathematics to mechanics by investigating equations for motion, force, energy, and other physical phenomena.
18.3 Imagine what an unsolvable equation might look like.

Geometry

Language of Geometry

1. Axiomatic Systems
    1.1   Distinguish between undefined terms and defined terms.
    1.2   Know which terms in geometric systems are often left undefined (e.g., point, line, plane, and intersect) and explain why they are often left undefined.
    1.3   Distinguish between postulates (axioms) and theorems (propositions).
    1.4   Know which statements in geometric systems are often assumed as postulates (e.g., through two points there exists exactly one straight line) and explain why they are often postulated.

2. Symbols
    2.1   Know and use basic geometric symbols, such as:

    A (Point A)

    $\overleftrightarrow{AB}$ (Line AB)

    $\overline{AB}$ (Segment AB)

    $\overrightarrow{AB}$ (Ray AB)

    $\overarc{AB}$ (Arc AB)

    $\angle$ (Angle)

    $\llcorner$ (Right Angle)

    $\triangle$ (Triangle)

    $\triangle ABC$ (Triangle ABC)

    $\measuredangle$ (Right Triangle)

    $\parallel$ (Parallel)

    $\perp$ (Perpendicular)

    $\cong$ (Congruent)

    $\sim$ (Similar)

    $\therefore$ (Therefore)

3. Constructions
    3.1   Understand a *construction* to be the drawing of a geometric figure using only two instruments, a compass and a straightedge.
    3.2   Draw the unique straight line between two points using a straightedge.
    3.3   Draw the unique circle given a center and a distance using a compass.
    3.4   Perform geometric constructions, such as:
        - Construct a segment congruent to a given segment with a given endpoint.
        - Construct an angle congruent to a given angle with a given side.
        - Construct the bisector of a given angle.
        - Construct the perpendicular bisector of a given segment.
        - Given a point on a line, construct the perpendicular to the line at the given point.
        - Given a point off a line, construct the perpendicular to the line from the given point.
        - Given a point off a line, construct the parallel to the given line through the given point.
        - Given a triangle, circumscribe a circle about the triangle.
        - Given a triangle, inscribe a circle within the triangle.
        - Given two segments, construct their geometric mean.

4. Logic, Reasoning, and Proof
    4.1   Recognize conditional (if-then) statements and identify the hypothesis and the conclusion.
    4.2   Distinguish between a conditional statement and a biconditional (if-and-only-if) statement.
    4.3   Form the inverse, converse, and contrapositive of a conditional statement.
    4.4   Develop conjectures using inductive reasoning.
    4.5   Disprove a statement by using a counterexample.
    4.6   Prove geometric statements using deductive reasoning.
    4.7   Give reasons for steps in a proof by appealing to given information, definitions, postulates/axioms, corollaries, and previously-proven theorems.
    4.8   Distinguish between direct proof and indirect proof (proof by contradiction) and use both kinds of reasoning to support arguments.
    4.9   Know that there may be multiple proofs for an individual theorem.

Figures and Theorems

5. Points, Lines, Planes, and Angles
    5.1   Understand a *point* to be a zero-dimensional object with location but no length, width, or height.

5.2   Understand a *line* to be a one-dimensional object with length but no width or height, extending indefinitely in opposite directions.

5.3   Understand a *plane* to be a two-dimensional object with length and width but no height, extending indefinitely in both dimensions.

5.4   Describe groups of points as collinear, noncollinear, coplanar, or noncoplanar.

5.5   Describe pairs of lines as parallel, perpendicular, intersecting, or skew.

5.6   Know the relationships between points, lines, and planes (e.g., two lines intersect at a point, two planes intersect at a line, exactly one plane can be drawn through a line and a point not on that line).

5.7   Understand a *segment* to be a part of a line between two endpoints.

5.8   Understand a *ray* to be a part of a line with one endpoint that continues indefinitely in one direction.

5.9   Understand an *angle* to be a figure formed by two rays (*sides*) with the same endpoint (*vertex*).

5.10   Measure angles in degrees with a protractor.

5.11   Describe angles as *acute* (between 0° and 90°), *right* (exactly 90°), *obtuse* (between 90° and 180°), and *straight* (exactly 180°) and reflex (between 180° and 360°).

5.12   Describe pairs of angles as congruent, incongruent, adjacent, vertical, supplementary, complementary, alternate interior, alternate exterior, and/or corresponding.

5.13   Prove and use theorems about angle pairs (e.g., when two lines intersect at a point, the pairs of vertical angles are congruent).

5.14   Solve for unknown angle measures using angle pair relationships.

5.15   Understand *bisection* to be a process that divides a figure into two congruent halves, and draw bisectors of segments and angles.

5.16   Prove and use theorems about bisection (e.g., any point lying on the bisector of an angle is equidistant from the sides of that angle).

6.   Polygons

6.1   Understand a *polygon* to be a closed two-dimensional figure formed by line segments (sides).

6.2   Distinguish between convex polygons and concave polygons.

6.3   Distinguish between regular and nonregular polygons.

6.4   Classify polygons according to their sides and their angles.

6.5   Understand a *diagonal* of a polygon to be a segment joining two nonconsecutive vertices of the polygon.

6.6   Conjecture, prove, and use the formula for the angle sum of a convex polygon with n sides $[(n-2)*180˚]$.

6.7   Understand *similar* polygons to be polygons whose corresponding angles are congruent and whose corresponding sides are proportional.

6.8   Solve for unknown side lengths and angle measures in similar polygons.

7. Triangles
    7.1   Understand a *triangle* to be a three-sided polygon.
    7.2   Identify, draw, and describe the properties of equilateral, isosceles, scalene, right, acute, and obtuse triangles.
    7.3   Conjecture, prove, and use theorems about triangles (e.g., the exterior angle theorem, the triangle inequality).
    7.4   Conjecture, prove, and use the various triangle congruences (SAS, ASA, SSS, AAS, and Hypotenuse-Leg).
    7.5   Know that the corresponding parts of congruent triangles are congruent, and solve for unknown side lengths and angle measures.
    7.6   Conjecture and prove that the sum of the angles of a triangle is 180°, and use it to find unknown angle measures of a triangle.
    7.7   Construct the medians of a triangle and know that their intersection is the triangle's centroid (center of mass).
    7.8   Know that the centroid is two-thirds of the distance from a vertex to the midpoint of the opposite side.
    7.9   Construct the altitudes of a triangle and know that their intersection is the triangle's orthocenter.
    7.10   Construct the perpendicular bisectors of the sides of a triangle and know that their intersection is the triangle's circumcenter.
    7.11   Construct the angle bisectors of a triangle and know that their intersection is the triangle's incenter.
    7.12   Know that the centroid, orthocenter, and circumcenter are collinear and that they define Euler's line.

8. Quadrilaterals
    8.1   Understand a *quadrilateral* to be a four-sided polygon.
    8.2   Identify, draw, and describe the properties of parallelograms, rhombuses, rectangles, squares, kites, and trapezoids.
    8.3   Conjecture, prove, and use theorems about quadrilaterals (e.g., the diagonals of a parallelogram bisect each other).

9. Right Triangles
    9.1   Understand a *right triangle* to be a triangle with one 90° (right) angle.
    9.2   Conjecture, prove, and use the Pythagorean theorem (in a triangle, $a^2 + b^2 = c^2$).
    9.3   Show that a triangle is a right triangle by using the converse of the Pythagorean theorem.
    9.4   Recall from memory common Pythagorean triples (3-4-5, 5-12-13, 8- 15-17, 7-24-25, etc.) and know that any multiple of a Pythagorean triple is also a Pythagorean triple.

9.5   Recall from memory angle and side relationships for special right triangles, including 30-60-90 (x, $\sqrt{3}x$, $2x$) and 45-45-90 (x, x, $\sqrt{2}x$) triangles.

9.6   Define and know the trigonometric ratios of sine ($\frac{opposite}{hypotenuse}$), cosine ($\frac{adjacent}{hypotenuse}$), and tangent ($\frac{opposite}{adjacent}$) in a right triangle.

9.7   Solve for unknown side lengths and angles in right triangles using trigonometric ratios.

9.8   **CHALLENGE TOPIC**: Prove the Pythagorean theorem using Euclid's method.

10. Circles

10.1   Understand a *circle* to be a set of points in a plane at a given distance (radius) from a given point (center).

10.2   Identify, draw, and describe the properties of a radius, diameter, chord, secant, and tangent of a circle.

10.3   Conjecture, prove, and use theorems about circles (e.g., the tangents to a circle from a point are congruent).

10.4   Identify, draw, and describe arcs, central angles, sectors, and semicircles.

10.5   Find the measures of arcs and central angles in degrees.

10.6   Identify, draw, and describe the properties of inscribed and circumscribed polygons and solve for unknown angle measures and side lengths.

10.7   **CHALLENGE TOPIC**: Know that as the number of sides increases of inscribed and circumscribed polygons with vertices on a circle, the ratio of the perimeter to the diameter of the circle for the inscribed and circumscribed polygons approaches π.

11. Solids

11.1   Understand a *solid* to be a closed three-dimensional figure.

11.2   Understand a *polyhedron* to be a solid bounded by faces, edges, and vertices.

11.3   Identify and describe the properties of prisms, pyramids, cylinders, cones, and spheres.

11.4   Conjecture, prove, and use theorems about solids (e.g., the length of a diagonal through a rectangular prism is equal to $\sqrt{l^2 + w^2 + h^2}$).

Measurement

12. Perimeter, Area, Volume, and Surface Area

12.1   Conjecture, prove, and use formulas for the perimeters of polygons, squares, parallelograms, and circles.

12.2   Conjecture, prove, and use formulas for the areas of triangles, squares, parallelograms, trapezoids, circles, and circle sectors.

12.3   Conjecture, prove, and use theorems about areas and perimeters (e.g., the area of a rhombus equals half the product of its diagonals).

12.4   Conjecture, prove, and use formulas for the volume and surface area for prisms, pyramids, cylinders, cones, and spheres.

12.5   Find the area, perimeter, volume, and surface area of composite figures and solids.

12.6   Know how changes in the length, width, or height of a figure or solid affect its perimeter, area, volume, and surface area.

13. Coordinate Geometry

13.1   Conjecture, prove, and use the midpoint formula ($\frac{x_1+x_2}{2}$, $\frac{y_1+y_2}{2}$) to find the midpoint of a segment on a coordinate plane.

13.2   Conjecture, prove, and use the distance formula ($d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$) to find the distance between two points on a coordinate plane.

13.3   Conjecture, prove, and use the general equation of a circle ($(x - a)^2 + (y - b)^2 = r^2$)

14. Transformations

14.1   Understand a *mapping* to be a correspondence from one set of points (the pre-image) to another set of points (the image).

14.2   Understand a *transformation* to be a one-to-one mapping in which each unique point in the pre-image corresponds to a unique point in the image.

14.3   Understand an *isometry* to be a transformation in which every segment in the pre-image is mapped to a congruent segment in the image.

14.4   Prove that reflections, translations, and rotations are isometries and find the coordinates of unknown points under these transformations.

14.5   Understand mathematical *dilation* to be a non-isometric transformation that produces a smaller or larger similar figure.

14.6   Distinguish between dilations that are expansions (increases in size) and contractions (decreases in size) and find the values of unknown side lengths under both types of dilations.

14.7   Identify and describe composite transformations.

14.8   Know that the inverse of a transformation maps the image back to the pre-image.

14.9   Know that a figure in a plane has symmetry if there is an isometry that maps the figure onto itself.

14.10 Distinguish between line symmetry, point symmetry, rotational symmetry, and translational symmetry.

Advanced Topics

15. Non-Euclidean Geometry

15.1   Know that the geometries of different spaces (spheres, saddles, etc.) often require a shift in terms, postulates/axioms, and theorems.

15.2   Know the different possible assumptions about parallelism that can be made (through a point not on a line, either 0, 1, or infinitely many parallel lines can be drawn) and the possible spaces that those assumptions describe.

15.3   Explore the effects that different assumptions about parallelism have on geometric properties (e.g., triangles on a sphere have an angle sum greater than 180°).

Mathematical Practices

16. Students should:
   16.1   *Solve word problems using mathematical concepts including*: distances, right trian-
          gles, angle measure, trigonometry, circles, areas and perimeters, similarity, solids, and
          transformations.
   16.2   *Use appropriate physical and conceptual tools including*: straightedges, compasses,
          rulers, protractors, set squares, the coordinate plane, diagrams, constructions,
          two-column reasoning tables, logical representations (e.g., Venn diagrams), formulas,
          and scientific calculators.
   16.3   *Understand and use appropriate mathematical language and terminology including*:
          undefined terms, defined terms, postulates, axioms, theorems, propositions, construc-
          tion, proof, inductive, deductive, direct proof, indirect proof, counterexample, condi-
          tional, biconditional, congruent, similar, mapping, and isometry.
   16.4   *Create representations of mathematical scenarios, problems, and processes including*:
          visual, symbolic, verbal, contextual, and physical representations of a given problem,
          geometric constructions, illustrations, two-dimensional shapes, three-dimensional
          solids, graphs on a coordinate plane, and various proof charts.
   16.5   *Reason about mathematical relationships and give evidence for conclusions by*: devel-
          oping conjectures about geometric ideas by noticing patterns, using a variety of
          reasoning and proving methods (e.g., direct, indirect), drawing diagrams to illustrate
          problem contexts, using symbols to describe phenomena, recognizing special types of
          general figures (e.g., right triangles as special triangles, rhombuses as special parallelo-
          grams), and translating between algebraic and geometric representations.
   16.6   *Communicate mathematical thinking by*: making informal arguments to support
          conjectures, making formal arguments in a variety of ways including two-column
          proofs, flow charts, and written paragraphs, providing physical examples of abstract
          geometric concepts, and explaining why an answer does or does not make sense.

History of Mathematics

17. Read and discuss true stories of famous mathematicians. Such as:
   17.1   Euclid and the codification of geometry into an axiomatic system.
   17.2   Plato, Archimedes, Euler, and the development and properties of Platonic and
          Archimedean solids.
   17.3   Leonhard Euler and the proof of Euler's Line.
   17.4   Carl Gauss, Janos Bolyai, Nikolai Lobachevsky, the history of Euclid's parallel postulate,
          and the development of non-Euclidean geometries.

18. Explore the nature and purpose of mathematics.

    18.1   Describe the reasons why every branch of mathematics must rest on a set of assumed axioms and undefined terms, and investigate the kinds of choices that need to be made when selecting axioms and undefined terms.

    18.2   Connect mathematics to philosophy by examining the ways in which logic and mathematics inform each other.

    18.3   Imagine the geometries of other spaces: e.g., sphere, saddle, bottle, 'flatland', macro-world, and micro-world.

Algebra II

Language of Algebra

1. Axiomatic Systems
   1.1   Know the axioms of real numbers including: reflexive, symmetric, transitive, commutative, associative, and distributive.
   1.2   Know the properties of real numbers including: addition property of equality, multiplication property of equality, multiplication property of zero, etc.
   1.3   Justify the reasons for the steps in a mathematical argument or proof by appealing to an axiom, a property, or a previous step.

2. Numbers
   2.1   Know the relationships between different sets of numbers including: natural (counting) numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, imaginary numbers, and complex numbers.
   2.2   Understand that a set of numbers is *closed* under an operation if performing that operation on any number in the set produces another number in the set.
   2.3   Know which sets of numbers are closed under which of the four basic arithmetic operations.

3. Expressions, Equations, and Inequalities
   3.1   Write expressions using zero, one, or more than one variable.
   3.2   Simplify expressions by using the order of operations and by applying properties of real numbers.
   3.3   Write algebraic equations and inequalities with one or more than one variable.
   3.4   Solve multi-step equations and inequalities with one variable using inverse operations.
   3.5   Solve an equation or inequality with more than one variable for a particular variable.
   3.6   Understand a *compound (combined) inequality* to be a statement that joins two or more inequalities using the words 'and' or 'or.'
   3.7   Solve and graph compound inequalities.
   3.8   Solve and graph absolute value equations and inequalities.

4. Functions and Relations

    4.1   Understand a *relation* to be a collection of ordered pairs determined by an expression, rule, law, or mapping.

    4.2   Understand a *function* to be a relationship or rule that assigns to each input (independent value) exactly one output (dependent value).

    4.3   Know that all functions are relations but not all relations are functions.

    4.4   Determine whether a relationship is a function given a graph (using the vertical-line test), an equation, or a table of values.

    4.5   Translate between various ways of representing functions: algebraic (equation/formula), visual (graph), numerical (table), and written/verbal (scenario).

    4.6   Distinguish between an independent variable and a dependent variable.

    4.7   Find the x-intercept(s) and y-intercept of a function from an equation, graph, or table.

    4.8   Find the domain and range of a function.

    4.9   Determine the intervals where a function is increasing, decreasing, constant, or at a critical point (maximum or minimum).

    4.10 Determine the intervals where a function is concave up, concave down, or at an inflection point.

    4.11 Sketch a function given a set of conditions (e.g., sketch a function with a root at $x = 2$, a critical point at $(4, 3)$, and that decreases from $x = 5$ to $x = 7$).

    4.12 Recognize and categorize different types of functions including: linear, quadratic, polynomial, exponential, logarithmic, and rational.

Expressions, Equations, & Functions

5. Imaginary and Complex Numbers

    5.1   Understand an *imaginary number* to be the product of any real number and the imaginary unit i where $i^2 = -1$.

    5.2   Understand a *complex number* to be a number of the form $a + bi$ where a and b are real numbers and i is the imaginary unit.

    5.3   Know that the complex numbers include both the real numbers (when $b = 0$) and the imaginary numbers (when $a = 0$).

    5.4   Form the complex conjugate of a complex number.

    5.5   Add, subtract, multiply, and divide complex numbers.

    5.6   Recognize the cyclical pattern of the powers of i.

6. Linear Functions

    6.1   Understand a *linear function* to be a relationship between two variables with a constant rate of change whose graph is a straight line on the coordinate plane.

6.2   Find the slope, the x-intercept, and the y-intercept of a linear function from a table, graph, or equation.

6.3   Mentally calculate the slope of a linear function given a table, graph, or equation.

6.4   Draw a graph of a line given the slope and a point on the line, two points, an equation, or a table.

6.5   Write the equation of a linear function in slope-intercept form ($y = mx + b$), standard form ($Ax + By = C$), and point-slope form [$y - y_1 = m(x - x_1)$].

6.6   Know that two lines are parallel when they have the same slope.

6.7   Know that two lines are perpendicular when the product of their slopes is -1, or the two lines are horizontal and vertical.

6.8   Find the equation of a line parallel or perpendicular to a given line through a prescribed point.

6.9   Solve and graph linear inequalities and systems of linear inequalities.

7. Polynomial Expressions

7.1   Understand a *polynomial* to be a sum of algebraic terms having variables, coefficients, exponents, and/or constants.

7.2   Know that the general form of a polynomial in one variable is $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$.

7.3   Classify polynomials by degree and number of terms.

7.4   Add, subtract, multiply, and divide polynomials by monomials.

7.5   Add, subtract, and multiply polynomials by polynomials.

7.6   Raise polynomials to a power by expanding and combining like terms.

7.7   Factor out the greatest common monomial factor of a polynomial.

7.8   Multiply conjugate binomials and factor a difference of squares.

7.9   Square a binomial and factor a perfect square trinomial.

7.10 Divide two polynomials using long division and synthetic division (where applicable).

7.11 Know and use the remainder and factor theorems to find solutions to polynomial equations of degree > 2.

7.12 Understand the *Fundamental Theorem of Algebra* (a polynomial of degree will have exactly complex roots), and find complex roots of polynomial functions.

7.13 **CHALLENGE TOPIC**: Factor sums and differences of two cubes.

8. Quadratic Equations and Functions

8.1   Understand a *quadratic equation* to be an equation with a polynomial of degree 2.

8.2   Know that the general form of a quadratic equation is , where a ≠ 0.

8.3   Solve quadratic equations by factoring.

8.4   Solve quadratic equations by graphing the function and looking for roots.

8.5   Solve quadratic equations by completing the square.

8.6   Derive the quadratic formula [$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$] and use it to solve quadratic equations.

8.7   Understand the *discriminant* of a quadratic equation to be $D = b^2 - 4ac$, and know how the value of the discriminant informs the number and nature of the equation's solutions.

8.8   Understand a *quadratic function* to be a polynomial function where the highest power of the independent variable is 2.

8.9   Graph quadratic functions by finding the vertex, axis of symmetry, y-intercept, and x-intercept(s).

8.10 Transform a quadratic function into the form $y - k = a(x - h)^2$ to determine the vertex and axis of symmetry of the resulting parabola.

8.11 Write equations for quadratic functions given some combination of the roots of the function (x-intercepts), three points, the vertex, axis of symmetry and/or the y-intercept.

9. Rational Expressions, Equations, and Functions

9.1   Understand a *rational expression* to be a quotient of polynomials.

9.2   Simplify rational expressions, including complex fractions, by the rules for adding, subtracting, multiplying, and dividing fractions.

9.3   Understand a *rational equation* to be an equation of rational expressions.

9.4   Solve rational equations by multiplying both sides by the least common denominator.

9.5   Understand a *rational function* to be the quotient of two polynomial functions.

9.6   Graph rational functions and describe their domains, vertical asymptotes, horizontal asymptotes, and removable discontinuities (holes).

10. Radical Expressions, Equations, and Functions

10.1 Understand a *radical expression* to be an expression containing a radical sign, radicand, and root index.

10.2 Know the square root of x^2 is the absolute value of x ($\sqrt{x^2} = |x|$).

10.3 Find the positive and negative roots of a radical and represent them with the symbol $\pm$.

10.4 Simplify radical expressions using properties of radicals.

10.5 Understand a *radical equation* to be an equation with a variable inside the radicand.

10.6 Solve radical equations by isolating the radical expression and raising it to a power.

10.7 Understand a *radical function* to be a function with a variable inside the radicand.

10.8 Graph radical functions and describe any restrictions on their domains and ranges.

11. Exponential and Logarithmic Equations and Functions

11.1 Simplify expressions with exponents, powers, and roots, including expressions with negative and rational exponents, by using the laws of exponents.

11.2 Understand a *logarithm* to be the power to which a base must be raised to get another number.

11.3   Derive and use the properties and laws of logarithms including the product law [$\log_b m\ n = \log_b m + \log_b n$], quotient law [$\log_b \frac{m}{n} = \log_b m - \log_b n$], power law [$\log_b m^k = k * log_b m$], and change of base law [$\log_b a = \frac{\log_c a}{\log_c b}$].

11.4   Know that the exponential and logarithmic processes are inverses of one another.

11.5   Understand an *exponential equation* to be an equation with a variable in an exponent.

11.6   Understand a *logarithmic equation* to be an equation with a variable in the argument of a logarithm.

11.7   Solve exponential and logarithmic equations using inverse operations.

11.8   Understand an *exponential function* to be a function with a variable in an exponent.

11.9   Understand a *logarithmic function* to be a function with a variable in the argument of a logarithm.

11.10   Graph exponential functions and find their y-intercepts, domains, ranges, and horizontal asymptotes.

11.11   Graph logarithmic functions and find their x-intercepts, domains, ranges, and vertical asymptotes.

12. Systems of Equations.

12.1   Understand a *system of equations* to be a set of two or more equations with the same variables.

12.2   Know that a solution to a system of equations is a set of values for the variables that makes all the equations true.

12.3   Distinguish between linear and non-linear systems of equations.

12.4   Solve linear and non-linear systems of equations with two variables by substitution, by combination, and by graphing the equations on a coordinate plane and looking for the point(s) of intersection.

12.5   Solve linear systems of equations with three variables.

Advanced Topics

13. Analytic Geometry

13.1   Derive and use the distance formula ($d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$) to find the distances between points on a coordinate plane.

13.2   Derive and use the equation of a circle ($(x - h)^2 + (y - k)^2 = r^2$) to find the circle's center and radius.

13.3   Derive and use the midpoint formula ($\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$) to find the midpoints of segments on a coordinate plane.

14. Trigonometry

14.1   Know that in a right triangle the measure of one of the acute angles relates to the ratios of the triangle's side lengths.

14.2 Understand the trigonometric functions *sine* ($\frac{opposite}{hypotenuse}$), *cosine* ($\frac{adjacent}{hypotenuse}$), and *tangent* ($\frac{opposite}{adjacent}$) and the specific ratios of side lengths to which they refer.

14.3 Solve for unknown sides of a right triangle using trigonometric relationships.

15. Sequences and Series

15.1 Understand a *sequence* as a list of numbers or objects in a particular order.

15.2 Understand an *arithmetic sequence* to be a sequence with a constant difference between any two successive terms and find the common difference of arithmetic sequences.

15.3 Understand a *geometric sequence* to be a sequence in which the ratio of any two successive terms is constant and find the ratio of geometric sequences.

15.4 Determine whether a sequence is arithmetic, geometric, or neither.

15.5 Describe a sequence's pattern and find missing terms in a sequence.

15.6 Understand a *series* to be an expression that adds together the terms of a sequence.

15.7 Distinguish between finite and infinite series.

15.8 Evaluate partial sums of a series.

Mathematical Practices

16. Students should:

16.1 *Solve word problems using mathematical concepts including*: absolute values, linear equations, quadratic equations, rational equations, radical equations, exponential and logarithmic equations, systems of equations, systems of inequalities, linear programming problems, trigonometry, and probability.

16.2 *Use appropriate physical and conceptual tools including*: the coordinate plane, variables, expressions, equations, tables, graphs, charts, diagrams, formulas, and graphing calculators.

16.3 *Understand and use appropriate mathematical language and terminology including*: closure, relation, critical point, maximum, minimum, increasing, decreasing, concave up, concave down, inflection point, asymptote, exponential, logarithmic, rational, trigonometric, discriminant, sequence, and series.

16.4 *Create representations of mathematical scenarios, problems, and processes including*: visual, symbolic, verbal, contextual, and physical representations of a given problem, Venn diagrams, graphs on a coordinate plane, various function representations, and sequence depictions.

16.5 *Reason about mathematical relationships and give evidence for conclusions by*: drawing diagrams to illustrate problem contexts, shifting fluidly between different representations (e.g., graph to equation), recognizing, describing, extending, and creating numerical patterns, deriving a formula by assigning variables, writing equations, simplifying, substituting, and rearranging to isolate a quantity of interest, and making connections between related problems, methods, and concepts.

16.6 *Communicate mathematical thinking by*: providing a justification for a given formula, explaining the steps of a solution process, identifying key moments and potential pitfalls in a mathematical method, relating solutions back to the context of the problem, discussing the relative merits of different solution methods, and explaining why an answer does or does not make sense.

History of Mathematics

17. Read and discuss true stories of famous mathematicians. Such as:
    17.1 John Napier and the use of logarithmic tables.
    17.2 The Pythagorean comma, the history of musical tuning, and logarithms.
    17.3 René Descartes and the use of algebraic equations to describe plane curves.
    17.4 Leonardo Fibonacci and the Fibonacci sequence.

18. Explore the nature and purpose of mathematics.
    18.1 Describe the ways in which mathematics can and cannot be conceived of as a language.
    18.2 Connect mathematics to chemistry by investigating chemical equations and the ways they must be balanced.
    18.3 Imagine the process of proof and how we go about proving mathematical relationships that we observe to be true.

Pre-Calculus

Language of Pre-Calculus

1. Numbers, Expressions, and Equations
   1.1 Know the relationships between different sets of numbers including: complex numbers, imaginary numbers, real numbers, irrational numbers, rational numbers, integers, whole numbers, and natural (counting) numbers.
   1.2 Simplify or evaluate numeric and algebraic expressions.
   1.3 Solve multi-step equations and inequalities.

2. Functions and Graphs
   2.1 Understand a *function* to be a relationship or rule that assigns to each input (independent value) exactly one output (dependent value).
   2.2 Translate between different conceptualizations of a function including a rule, a relationship, a mapping, an equation, and a graph.
   2.3 Know the terminology of functions including domain, codomain, range, independent variable, dependent variable, x-intercept, y-intercept, root, critical point, maximum, minimum, increasing, decreasing, constant, concave up, concave down, inflection point, and asymptote.
   2.4 Translate between various ways of representing functions: algebraic (equation/ formula), visual (graph), numerical (table), and written/verbal (scenario).
   2.5 Recognize and categorize different types of functions including: constant, linear, quadratic, polynomial, radical, exponential, logarithmic, rational, and trigonometric.
   2.6 Sketch a function given a set of conditions (e.g., sketch a function with a root at $x = -3$, a critical point at (1, 6), an inflection point at (3, 2), and that decreases from $x = 1$ to $x = 5$).
   2.7 Add, subtract, multiply, and divide functions.
   2.8 Recognize, graph, and analyze function transformations, including horizontal stretches (dilations), reflections, and shifts (translations) and vertical stretches (dilations), reflections, and shifts (translations).
   2.9 Know that a function is even when it is symmetric with respect to the y-axis and odd when it is symmetric with respect to the origin.

2.10   Determine algebraically and graphically whether a function is even, odd, or neither.

2.11   Understand function *composition* to be the application of one function to the results of another function.

2.12   Find the composition of multiple given functions and determine the domain of the composition.

2.13   Understand the *inverse* of a function to be the correspondence from the function's range back to its domain.

2.14   Verify two functions are inverses of one another by showing that a composite of the two functions is the identity function [f(x) = x].

2.15   Find the inverse of a given function and determine the domain and range of the inverse.

2.16   Determine if a function is a one-to-one function by applying the horizontal line test.

2.17   Know that a function must be a one-to-one function in order for its inverse to be a function,

2.18   Graph and analyze piecewise defined functions including absolute value functions and functions with multiple formulas and domain intervals.

Equations & Functions

3.   Linear, Polynomial, and Rational Equations and Functions

3.1   Graph and analyze linear functions, including finding their slope and converting between standard form, slope-intercept form, and point-slope form.

3.2   Graph and analyze quadratic functions, including finding the vertex and intercepts, and converting between standard form and transformation form.

3.3   Know and use the remainder and factor theorems to find solutions to polynomial equations of degree > 2.

3.4   Know and use the Fundamental Theorem of Algebra (a polynomial of degree n will have exactly n complex roots), and find complex roots of polynomial functions.

3.5   Graph and analyze polynomial functions by applying the Leading Coefficient Test, finding zeros and critical points, and identifying end behavior.

3.6   Graph and analyze rational functions, including finding their holes (removable discontinuities), vertical asymptotes, and horizontal or oblique asymptotes.

4.   Irrational, Exponential, and Logarithmic Equations and Functions

4.1   Graph and analyze irrational functions, including finding any restrictions on the domain and range.

4.2   Understand a *logarithm* to be the power to which a base must be raised to get another number.

4.3   Derive and use the properties and laws of logarithms including the product law [$\log_b m \quad n = \log_b m + \log_b n$], quotient law [$\log_b \frac{m}{n} = \log_b m - \log_b n$], power law [$\log_b m^k = k * log_b m$], and change of base law [$\log_b a = \frac{\log_c a}{\log_c b}$].

4.4   Derive a formula for Euler's number (*e*), and know that it is a transcendental, irrational number approximately equal to $2.71828\ldots$.

4.5   Know that the natural logarithm function (*ln*) is the inverse of the exponential function with base e.

4.6   Solve exponential and logarithmic equations.

4.7   Graph and analyze an exponential function, including finding its y-intercept and horizontal asymptote.

4.8   Graph and analyze a logarithmic function, including finding its x-intercept and vertical asymptote, and describing the restriction in its domain.

4.9   Verify an exponential function and a logarithmic function are inverses of one another.

4.10 Find the inverses of exponential and logarithmic functions.

4.11 Apply algebraic and geometric transformations to graph and analyze radical, exponential, and logarithmic functions.

Trigonometry

5.   Triangle Trigonometry

5.1   Know that in a right triangle the measure of one of the acute angles relates to the ratios of the triangle's side lengths.

5.2   Understand the trigonometric functions sine ($\frac{opposite}{hypotenuse}$), cosine ($\frac{adjacent}{hypotenuse}$), and tangent ($\frac{opposite}{adjacent}$), *cotangent* ($\frac{adjacent}{opposite}$), *secant* ($\frac{hypotenuse}{adjacent}$), and *cosecant* ($\frac{hypotenuse}{opposite}$), and know the specific ratios of side lengths to which they refer.

5.3   Find values of the six trigonometric functions of a given angle.

5.4   Derive and use the law of sines ($\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$).

5.5   Derive and use the law of cosines ($c = \sqrt{\ +\ -\ 2ab * \cos\gamma}$).

5.6   Define inverse trigonometric functions as functions that map a trigonometric ratio to an angle.

5.7   Solve for the sides and angles of any given triangle by using trigonometric relationships and laws.

5.8   Derive and use the trigonometric formula for the area of a triangle ($A = \frac{1}{2}ab * sin\gamma$).

5.9   **CHALLENGE TOPIC**: Find the area of a triangle using Heron's Formula ($A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$).

6.   The Unit Circle and Angle Measure

6.1   Understand the *unit circle* to be the circle with center (0, 0) and radius 1.

6.2   Understand a *radian* to be a unit of measure equal to the measure of the central angle of a circle that subtends an arc equal in length to the radius of that circle (~57.3˚).

6.3   Measure an angle in radians by finding the ratio between the length of the arc it subtends and its radius.

6.4   Measure angles in degrees and radians including angles in standard position (between 0° and 360° / between 0 and 2π radians), angles greater than 360° (2π radians), and angles less than 0° (0 radians).

6.5   Convert between degree and radian measures.

6.6   Recall from memory the values of the six trigonometric functions at 0°, 30°, 45°, 60°, and 90° (0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ radians).

6.7   Define and find the values of the six trigonometric functions in all four quadrants by drawing reference triangles and determining the correct positive or negative sign.

6.8   Derive and use formulas for arc length ($s = r\theta$), sector area ($A = \frac{1}{2}r^2\theta$), and angular velocity ($\omega = \frac{\theta}{t}$).

7. Trigonometric Graphs

7.1   Define the six trigonometric functions using the unit circle, namely that if x, y is a point on the unit circle corresponding to angle θ, then $sin\ \theta = y$, $cos\ \theta = x$, $tan\ \theta = \frac{y}{x}$, $cot\ \theta = \frac{x}{y}$, $sec\ \theta = \frac{1}{x}$, and $csc\ \theta = \frac{1}{y}$.

7.2   Understand a *periodic function* to be a function that repeats its values at regular intervals or periods, satisfying the equation $f(x + p) = f(x)$ for some constant p.

7.3   Graph the six trigonometric functions and find their domains, ranges, intercepts, periods, amplitudes, and asymptotes.

7.4   Draw and analyze transformations of trigonometric functions, including period, amplitude, and phase shift.

7.5   Graph the six inverse trigonometric functions with restricted domains and ranges.

8. Trigonometric Identities

8.1   Derive and use basic trigonometric identities, including the reciprocal identities, the cofunction identities, and the Pythagorean identities.

8.2   Derive and use sum and difference trigonometric formulas (e.g., $\sin(a + b) = sin\ a\ cos\ b + cos\ a\ sin\ b$).

8.3   Derive and use double-angle and half-angle trigonometric formulas (e.g., $\sin\ 2a = 2\ sin\ a\ cos\ a$).

8.4   Derive and use sum-to-product and product-to-sum trigonometric formulas (e.g., $sin\ a + sin\ b = 2\left[sin\left(\frac{a+b}{2}\right)cos\left(\frac{a-b}{2}\right)\right]$).

8.5   Verify other trigonometric identities.

8.6   Solve trigonometric equations using inverse trigonometric functions and trigonometric identities, checking for extraneous solutions.

8.7   Solve trigonometric equations in quadratic form.

9. Trigonometric Applications: Polar Coordinates and Complex Numbers

9.1   Understand the *polar coordinate system* as a coordinate system in which points are described by the ordered pair (r, θ) where r is the radial distance from the origin and θ is the angle between the positive x-axis and the line connecting the point to the origin.

9.2   Convert between polar coordinates and Cartesian coordinates.

9.3   Convert equations between polar form and rectangular form.

9.4   Graph and classify polar functions (e.g., lines, circles, cardioids, limaçons, lemniscates, roses, and spirals).

9.5   Understand the *complex plane* as a way to represent complex numbers.

9.6   Convert between the rectangular form ($z = x + iy$) and the polar (trigonometric) form ($z = r(cos\theta + isin\theta)$) of complex numbers.

9.7   Multiply and divide complex numbers in polar form.

9.8   Derive and use De Moivre's Theorem to find the nth power of a complex number.

9.9   Find the nth root of complex numbers.

10. Trigonometric Applications: Vectors

10.1   Understand a *vector* to be a directed line segment with a magnitude and direction that is independent of the coordinate system.

10.2   Find the vector from an initial point to a terminal point in a coordinate plane.

10.3   Add and subtract vectors geometrically and arithmetically.

10.4   Multiply a vector by a scalar.

10.5   Find the magnitude and direction of a given vector.

10.6   Find a unit vector of a given vector.

10.7   Find the dot product of a pair of vectors.

10.8   Find the angle between two vectors.

10.9   **CHALLENGE TOPIC**: Perform operations on three-dimensional vectors.

10.10 **CHALLENGE TOPIC**: Find the cross product of three-dimensional vectors.

10.11 **CHALLENGE TOPIC**: Find the direction angles of a vector.

Advanced Topics

11. Analytic Geometry

11.1   Know that the four different conic sections—circles, parabolas, ellipses, and hyperbolas—are formed by different slices of a cone.

11.2   Derive and use the equation of a circle [$(x - h)^2 + (y - k)^2 = r^2$] to find the circle's center and radius.

11.3   Derive and use the equation of a parabola [$y - k = a(x - k)^2$] to find the parabola's vertex, focus, directrix, and axis of symmetry.

11.4   Derive and use the equation of an ellipse ($\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$) to find the ellipse's center, semi-major axis, semi-minor axis, and foci.

11.5   Derive and use the equation of a hyperbola ($\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$) to find the hyperbola's center, transverse axis, conjugate axis, foci, and asymptotes.

11.6   Describe the eccentricity of each of the four conic sections.

11.7   Convert the polar equation of a conic to a rectangular equation.

11.8   Graph and analyze polar equations of conics.

11.9   **CHALLENGE TOPIC**: Understand a *parametric equation* to be a relationship where both the dependent variable y and the independent variable x are written as functions of a third variable t (called a parameter).

11.10   **CHALLENGE TOPIC**: Sketch a parametric curve by creating a table of values and by using a graphing calculator.

12. Sequences and Series

12.1   Understand a *sequence* as a list of numbers in a particular order and as a function with a domain of positive integers.

12.2   Write sequences with explicit and recursive formulas.

12.3   Understand a *series* to be an expression that adds together the terms of a sequence.

12.4   Write series using sigma notation, and identify the index and summand.

12.5   Distinguish between arithmetic and geometric sequences and series.

12.6   Distinguish between finite and infinite sequences and series.

12.7   Distinguish between infinite series that converge and ones that diverge.

12.8   Evaluate partial sums of a series.

12.9   Derive and use the formula for the sum of a geometric series $(ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r})$.

12.10   **CHALLENGE TOPIC**: Show a summation formula to be true using the principle of mathematical induction.

12.11   **CHALLENGE TOPIC**: Verify the continued fraction expansions of irrational numbers (e.g., $\sqrt{2}$, φ, e).

13. Systems of Equations and Matrices

13.1   Solve systems of linear equations in two, three, or more variables by graphing, substitution, or combination.

13.2   Solve non-linear systems of equations in two, three, or more variables by graphing, substitution, or combination.

13.3   Understand a *matrix* to be a rectangular array of numbers enclosed by brackets or parentheses.

13.4   Determine the dimension of a matrix by counting the rows and columns.

13.5   Add and subtract matrices with the same dimensions.

13.6   Multiply or divide a matrix by a scalar.

13.7   Multiply two matrices of compatible dimensions.

13.8   Find the determinant of a $2x2$ or a $3x3$ matrix.

13.9   Understand the *identity matrix* to be $\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$.

13.10   Find the inverse of a given square matrix with a non-zero determinant.

13.11   Represent a system of linear equations with an augmented matrix.

13.12 Solve a system of linear equations by performing row operations to obtain the reduced row echelon form of the matrix.

13.13 Solve a system of linear equations using the determinant method (Cramer's Rule).

13.14 Solve a system of linear equations using inverse matrices.

13.15 **CHALLENGE TOPIC**: Find the area and volume of 2-dimensional and 3-dimensional figures using the geometric properties of determinants.

13.16 **CHALLENGE TOPIC**: Write the partial fraction decomposition of a given rational expression using a system of equations and matrix operations.

13.17 **CHALLENGE TOPIC**: Construct the adjacency matrix of a connected graph to find the distances between two vertices in order to know how ranking algorithms, such as PageRank, order items based on relevance or importance.

Probability and Statistics

14. Combinatorics

14.1 Understand a *factorial* (symbolized by !) to be the product of an integer and all the positive integers below it, and calculate factorials of integers.

14.2 Derive and use basic counting principles, including the multiplication rule, the addition rule, and the complement rule.

14.3 Understand a *permutation* to be an arrangement of objects in which order matters.

14.4 Derive and use the formula for a permutation ($nPr = \frac{n!}{(n-r)!}$).

14.5 Understand a *combination* to be an arrangement of objects in which order does not matter.

14.6 Derive and use the formula for a combination ($nCr = \frac{n!}{r!(n-r)!}$).

14.7 Count the number of possible arrangements of a group of objects using permutations and combinations.

14.8 Perform binomial expansions using both Pascal's triangle and the Binomial Theorem.

15. Probability

15.1 Understand *probability* as the likelihood or chance of an event occurring.

15.2 Calculate the probabilities of mutually exclusive, overlapping (dependent), and independent events.

15.3 Derive and use the Binomial Probability Theorem (the probability of k successes in n trials of an event given a success probability p for each trial is $nCk * p^k * \left(1 - p\right)^{n-k}$).

15.4 **CHALLENGE TOPIC**: Understand *conditional probability* to be the probability that an event will happen given the knowledge that another event has already happened.

15.5 **CHALLENGE TOPIC**: Derive and use the conditional probability formula $P\left(B\middle|A\right) = \frac{P(A \ and \ B)}{P(A)}$.

16. Statistics

16.1 Represent a data set with a frequency table, a histogram, a stem and leaf plot, and a box plot.

16.2 Calculate different measures of central tendency: mean, median, and mode.

16.3 Calculate different measures of dispersion: range, variance, standard deviation, and interquartile range.

16.4 Represent a set of bivariate data with a scatter plot.

16.5 Describe patterns in the data such as clustering, outliers, positive and negative association, and linear and nonlinear association.

16.6 Fit a straight line to a scatter plot.

16.7 Understand the *normal distribution* to be a bell-shaped curve that is symmetric about its mean, and identify the kinds of data that might have a normal distribution (e.g., height, birth weight, job satisfaction).

16.8 **CHALLENGE TOPIC**: Understand the *standard normal distribution* to be a normal curve with a mean of 0 and a standard deviation of 1, described by the equation $y = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.

Mathematical Practices

17. Students should:

17.1 *Solve word problems using mathematical concepts including*: functions, (linear, quadratic, rational, radical, exponential, logarithmic and trigonometric), vectors, polar coordinates, trigonometric equations, force, work, energy, combinatorics, probability, statistics, and matrices.

17.2 *Use appropriate physical and conceptual tools including*: the coordinate plane (Cartesian and polar), variables, expressions, equations, tables, graphs, charts, diagrams, formulas, statistical models, matrices, and graphing calculators.

17.3 *Understand and use appropriate mathematical language and terminology including*: imaginary numbers, codomain, even and odd functions, composition, inverse, radians, degrees, periodicity, unit circle, vector, scalar, sigma notation, factorial, expected value, normal distribution, matrix, and limit.

17.4 *Create representations of mathematical scenarios, problems, and processes including*: visual, symbolic, verbal, contextual, and physical representations of a given problem, the unit circle, graphs on the Cartesian coordinate plane, various representations of functions, the polar coordinate system, the complex plane, matrices, probability trees, frequency tables, histograms, stem and leaf plots, box plot, and scatter plots.

17.5 *Reason about mathematical relationships and give evidence for conclusions by*: drawing diagrams to illustrate problem contexts, shifting fluidly between different representations (e.g., graph to equation), relating algebraic concepts and formulas to their geometric representation, and applying mathematical concepts to physical scenarios.

17.6   *Communicate mathematical thinking by*: discussing the differences between formal and informal definitions of mathematical concepts (e.g., functions), describing the ways in which certain mathematical concepts (e.g., limits) depend on some understanding of the infinite or the indefinite, explaining the steps of a solution process, relating solutions back to the context of the problem, and explaining why an answer does or does not make sense.

History of Mathematics

18. Read and discuss true stories of famous mathematicians. Such as:
   18.1   Apollonius of Perga and the discovery of conic sections.
   18.2   Hipparchus and the exploration of trigonometry.
   18.3   Leonhard Euler, Augustin-Louis Cauchy, August Ferdinand Möbius and the expression 0^0.
   18.4   James Sylvester, Arthur Cayley, and the development of matrices.

19. Explore the nature and purpose of mathematics.
   19.1   Describe the relationship between algebra and geometry.
   19.2   Connect mathematics to architecture by investigating the ways in which building design and construction rely on trigonometry, geometry, and other mathematical principles.
   19.3   Imagine the branches of mathematics that are built upon Pre-Calculus, including Calculus, Linear Algebra, Statistics, Probability, and Number Theory.

Calculus

Language of Calculus

1. Nature of Calculus
    1.1   Understand *calculus* to be a branch of mathematics that applies the idea of infinitesimal quantities to the study of continuous change.
    1.2   Know that calculus centers on two related (and inverse) ideas, the derivative and the integral.
    1.3   Know that both the derivative and the integral rely upon the idea of a limit, a rigorous formulation of an infinitesimal.

2. Motivations
    2.1   Recognize the "area problem" and the method of exhaustion used to approximate the area of curved shapes by inscribing and circumscribing polygons with more and more sides.
    2.2   Recognize the "tangent problem" and the method of finding the slope of the line tangent to a function at a point by finding the slope of a secant line and bringing the two points closer and closer together.
    2.3   Recognize the "instantaneous velocity problem" and the method of calculating the average velocity over smaller and smaller intervals.
    2.4   Recognize the "infinite series problem" and the method of adding up more and more terms to approach the sum of the series.

3. Functions
    3.1   Understand a *function* to be a relationship or rule that assigns to each input (independent value) exactly one output (dependent value).
    3.2   Know the terminology of functions including domain, codomain, range, independent variable, dependent variable, and be familiar with $f(x)$ functional notation.
    3.3   Translate between various ways of representing functions: algebraic (equation/ formula), visual (graph), numerical (table), and written/verbal (scenario).
    3.4   Know the terminology of the behavior of a function including: x-intercept, y-intercept, root, critical point, maximum, minimum, increasing, decreasing, constant, concave up, concave down, inflection point, and asymptote.

3.5   Recognize, graph and analyze different types of functions, including linear, quadratic, polynomial, radical, exponential, logarithmic, rational, and trigonometric.

3.6   Recognize, graph, and analyze function transformations, including horizontal stretches (dilations), reflections, and shifts (translations) and vertical stretches (dilations), reflections, and shifts (translations).

3.7   Recognize, graph and analyze combinations of functions, including function addition, subtraction, multiplication, division, composition, inverses, and piecewise defined functions.

Limits and Continuity

4.   Limits

4.1   Understand the informal definition of a *limit*, namely that $\lim\limits_{x \to a} f\left(x\right) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to (but not equal to) a.

4.2   Evaluate the limit of a function at a point by creating and analyzing a table of values.

4.3   Evaluate the limit of a function at a point by analyzing the graph of the function.

4.4   Evaluate one sided limits (left and right hand) by using a table of values or by analyzing the function's graph.

4.5   Evaluate limits at infinity and use them to describe horizontal asymptotes

4.6   Evaluate infinite limits and use them to describe vertical asymptotes.

4.7   Derive and use various laws and techniques for evaluating limits including the sum/difference law, the product/quotient law, the constant multiple law, the power law, the root law, algebraic manipulation, and the squeeze theorem.

4.8   **CHALLENGE TOPIC**: Understand the formal (epsilon-delta) definition of a *limit*, namely that $\lim\limits_{x \to a} f\left(x\right) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

5.   Continuity

5.1   Understand (informally) that a function is *continuous* over an interval when it can be drawn in that interval as a curve with no missing points, breaks, or jumps.

5.2   Understand (formally) that a function is *continuous* at a point when the value of the function at that point is equal to the limit (both left and right hand) of the function as x approaches that point ($\lim\limits_{x \to a} f\left(x\right) = f\left(a\right)$).

5.3   Know that a function is continuous over an interval when it is continuous at all the points within that interval.

5.4   Distinguish between a removable discontinuity, an infinite discontinuity, and a jump discontinuity.

5.5   Derive and use the Intermediate Value Theorem and the Extreme Value Theorem.

Differentiation

6. The Derivative
    6.1   Understand different interpretations of the *derivative* as the slope of a tangent line and as the instantaneous rate of change.
    6.2   Sketch a visual interpretation of the derivative on a graph, showing the secant line between two points approaching a tangent line as the distance between the two points approaches 0.
    6.3   Understand the *derivative* of a function f at a number a, denoted by $f\,'(a)$, to be $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ if the limit exists.
    6.4   Understand the *derivative* as a function $f\,'(x)$ to be $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$ if the limit exists.
    6.5   Recognize notations for the derivative, including $f\,'(x) = y\,' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f\left(x\right)$.
    6.6   Know when a given function is differentiable.
    6.7   Know that all differentiable functions are continuous, but not all continuous functions are differentiable (e.g., $f(x) = |x|$).

7. Calculating Derivatives
    7.1   Derive and use differentiation rules for combinations of functions including the constant multiple rule ($\frac{d}{dx}\left[c*f(x)\right] = c*f\,'(x)$), the sum/difference rule ($\frac{d}{dx}[f(x) \pm g(x)] = f\,'(x) \pm g\,'(x)$), the product rule ($\frac{d}{dx}[f(x)*g(x)] = f\,'(x)*g(x) + f(x)*g\,'(x)$), the quotient rule ($\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)*f\,'(x)-f(x)*g\,'(x)}{[g(x)]^2}$), and the chain rule [$\frac{d}{dx}f(g(x)) = f\,'\big(g((x))*g\,'(x)\big]$.
    7.2   Derive and use differentiation formulas for constant, linear, polynomial, trigonometric, inverse trigonometric, exponential, and logarithmic functions.
    7.3   Know and use implicit differentiation to find the derivatives of functions defined by equations.
    7.4   Calculate higher order derivatives, including $f\,''(x)$ and $f\,'''(x)$.

8. Derivative Applications
    8.1   Apply the basic idea of a derivative as an instantaneous rate of change to solve problems in physics (e.g., velocity, acceleration, current), chemistry (e.g., instantaneous rate of reaction, compressibility), biology (e.g., instantaneous growth rate), economics (e.g., marginal cost, marginal profit), and other fields.
    8.2   Create a linear approximation of a function and apply these linearizations to various contexts.
    8.3   Solve related rate problems in a variety of contexts.
    8.4   Solve optimization problems in a variety of contexts.
    8.5   Derive and use Rolle's Theorem and the Mean Value Theorem.
    8.6   Derive and use L'Hôpital's Rule.

8.7   Know the relationship between $f\,'(x)$, $f\,''(x)$, and the shape of $f(x)$, and use differentiation to sketch the graph of a function, identifying maxima, minima, inflection points, and intervals in which the function is increasing, decreasing, concave up, or concave down.

8.8   **CHALLENGE TOPIC**: Approximate the roots of a function using Newton's method.

Integration

9. The Integral

9.1   Understand that a function $F(x)$ is an *antiderivative* of $f(x)$ if $F\,'(x)$.

9.2   Know that if $F(x)$ is an antiderivative of $f(x)$ over an interval, then the most general antiderivative of $f(x)$ is $F(x) + C$ where C is an arbitrary constant.

9.3   Understand the *definite integral* to be a Riemann sum approximating the area under the curve of a function where the number of equal-width rectangles approaches infinity.

9.4   Sketch a visual interpretation of the definite integral.

9.5   Know the terminology and notation of definite integrals including the integral sign, integrand, limits of integration, upper bound, and lower bound.

9.6   Derive and use the Fundamental Theorem of Calculus relating the definite integral to the antiderivative of the function, that is, $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

9.7   **CHALLENGE TOPIC**: Distinguish between functions that are integrable and ones that are not (e.g., $\int \frac{\sin x}{x}dx$).

10. Calculating Integrals

10.1   Derive and use antidifferentiation formulas for constant, linear, polynomial, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

10.2   Derive and use antidifferentiation strategies for combinations of functions including the constant multiple rule, the sum/difference rule, u-substitution, integration by parts, trigonometric substitution, partial fraction decomposition, and the use of integration tables.

10.3   Approximate integrals that do not have closed-form antiderivatives using Simpson's rule.

10.4   Recognize and evaluate improper integrals as limits of definite integrals, and use a limit to test for convergence or divergence.

11. Integral Applications

11.1   Find the area under a curve and the area between two curves.

11.2   Find the volumes of solids by slicing, the disk method, and the cylindrical shell method.

11.3   Solve problems such as those involving velocity, acceleration, work, arc length, area of a surface of revolution, hydrostatic pressure and force, moments of inertia and centers of mass, consumer surplus, flux, and probability density functions.

11.4   Derive and use the Mean Value Theorem for integrals.

Advanced Topics

12. Differential Equations

12.1   Understand a *differential equation* to be an equation that contains an unknown function and one or more of its derivatives.

12.2   Know that the order of a differential equation is the order of the highest derivative that occurs in the equation.

12.3   Derive and use basic techniques for solving differential equations including slope fields, Euler's method, and separation of variables.

12.4   Find a particular solution to a linear differential equation using a general solution and a set of initial conditions.

12.5   Apply solution methods for linear differential equations to various models including population growth and decay, simple harmonic oscillator, mixing problems, logistic models, circuits, and predator-prey systems.

13. Infinite Sequences and Series

13.1   Understand an *infinite sequence* to be a list of numbers that continues indefinitely without a final term.

13.2   Write a formula for a general term of a_n of an infinite sequence.

13.3   Distinguish between a convergent and divergent infinite sequence, and use a limit to test for convergence or divergence.

13.4   Describe sequences as increasing, decreasing, monotonic, bounded above, bounded below, or unbounded, and know the relationships between these qualities of a sequence and whether it converges.

13.5   Understand an *infinite series* to be a sum of infinitely many numbers and know the relationship between an infinite series and the sequence of its partial sums.

13.6   Write a formula for an infinite series in sigma notation.

13.7   Derive and use the geometric series sum formula, and know when a geometric series diverges or converges.

13.8   Distinguish between a convergent and divergent infinite series and determine whether a given series diverges or converges by using convergence tests such as the divergence test, comparison test, limit comparison test, ratio test, root test, integral test, and alternating series test.

13.9   Understand a *power series* to be a series of the form
$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$, and determine the interval of convergence for a power series representation.

13.10 Differentiate and integrate a power series term-by-term.

13.11 Derive and use the formula for the Taylor series expansion of a function $f(x)$ centered at a value $x = a$, namely $f\left(a\right) + \frac{f'(a)}{1!}\left(x - a\right) + \frac{f'''(a)}{3!}\left(x - a\right)^3 + \cdots$.

13.12 Understand a *Maclaurin series* to be a Taylor series centered at 0, and verify the Maclaurin series expansion for common functions including e^x, $\sin x$, $\cos x$, $\tan^{-1}$, and $\frac{1}{1-x}$.

13.13 Find the nth-degree Taylor polynomial of a given function centered at a point and estimate the error of an approximation using Taylor's Inequality.

14. Parametric Equations and Polar Coordinates

14.1   Understand a *parametric equation* to be a relationship where both the dependent variable y and the independent variable x are written as functions of a third variable t (called a parameter).

14.2   Sketch a parametric curve by creating a table of values and by using a graphing calculator.

14.3   Apply the techniques of differentiation and integration to parametric curves to find tangents, areas, arc lengths, and surface areas.

14.4   Understand the *polar coordinate system* as an alternative to the Cartesian coordinate system in which points are described by the ordered pair (r, θ) where r is the radial distance from the origin and θ is the angle between the positive x-axis and the line connecting the point to the origin.

14.5   Sketch polar curves by creating a table of values and by using a graphing calculator.

14.6   Apply the techniques of differentiation and integration to polar curves (changing variables when appropriate) to find tangents, areas, and arc lengths.

Mathematical Practices

15. Students should:

15.1   Solve word problems using mathematical concepts including: various applications of derivatives and integrals.

15.2   *Use appropriate physical and conceptual tools including*: the coordinate plane (Cartesian and polar), variables, function models, diagrams, and graphing calculators.

15.3   *Understand and use appropriate mathematical language and terminology including*: limits, tangent, piecewise function, continuity, discontinuity, derivative, implicit differentiation, antiderivative, integral, optimization, differential equation, convergent, divergent, and parametric equation.

15.4 *Create representations of mathematical scenarios, problems, and processes including*: visual, symbolic, verbal, contextual, and physical representations of a given problem, various representations of functions, graphical depictions of limits and derivatives, and Riemann sums.

15.5 *Reason about mathematical relationships and give evidence for conclusions by*: modeling a concrete relationship with a mathematical function, drawing diagrams to illustrate problem contexts, selecting appropriate strategies that apply to a particular problem (e.g., selecting a convergence test, selecting an antidifferentiation strategy), and selecting the most appropriate representation of a mathematical object given a particular context (e.g., the various representations of the derivative $f'(x) = y' = \frac{dy}{dx}$).

15.6 *Communicate mathematical thinking by*: identifying common underlying structures in problems involving different contextual situations, explaining how an approximation relates to the actual value and discussing the merits of both, describing the ways in which certain mathematical concepts (e.g., limits) depend on some understanding of the infinite or the indefinite, explaining the steps of a solution process, relating solutions back to the context of the problem, and explaining why an answer does or does not make sense.

History of Mathematics

16. Read and discuss true stories of famous mathematicians. Such as:
    16.1 Isaac Newton, Gottfried Leibniz, and the parallel discovery of calculus.
    16.2 Johann Bernoulli and the discovery of L'Hôpital's Rule.
    16.3 Evangelista Torricelli and Gabriel's Horn.
    16.4 Leonhard Euler and Euler's identity ($e^{i\pi} + 1 = 0$).

17. Explore the nature and purpose of mathematics.
    17.1 Describe the ways in which calculus allows us to conceptualize the relationship between the discrete and the continuous.
    17.2 Connect mathematics to engineering by investigating the mathematical optimization of various structures and systems.
    17.3 Imagine what calculus would look like with more than two variables.

Statistics

Language of Statistics

1. Nature of Statistics
   1.1 Understand *statistics* to be a branch of mathematics that involves the collection, representation, and analysis of quantitative data.
   1.2 Know that statistical analysis draws upon the presumption that a random sample of a population will be representative of that population.
   1.3 Know that descriptive statistics summarizes and describes the main features of a data set.
   1.4 Know that inferential statistics infers characteristics of a population based on sample data.

2. Data
   2.1 Understand *data* to be a set of observations or measurements collected or produced as a source of information about an individual or a population.
   2.2 Know that data can be numbers, words, images, sounds, or other basic units of meaning.
   2.3 Distinguish between nominal, ordinal, discrete, and continuous data.
   2.4 Understand a *variable* to be a label given to an attribute or characteristic that can take different values for different individuals.
   2.5 Distinguish between categorical variables and quantitative variables.
   2.6 Distinguish between explanatory (independent) variables and response (dependent, outcome) variables.
   2.7 Understand two variables to be *associated* if knowing the value of one variable helps predict the value of the other.

3. Studies
   3.1 Understand a *study* to be an analysis of data in order to learn about a population.
   3.2 Understand an *observational study* to be a study that measures variables of interest in individuals without attempting to influence the responses.

3.3   Understand an *experiment* to be a study that deliberately imposes some treatment on individuals to measure their responses.

3.4   Know the terminology of an experiment, including placebo, treatment, subjects, factors, levels, control group, experimental group, double-blind, single-blind, and replication.

3.5   Distinguish between different study design methods, including completely random design, randomized block design, and matched pairs design (quasi-experimental design).

3.6   Understand the findings of a study to be statistically *significant* when the association between variables cannot be plausibly explained by chance.

3.7   Know that statistical significance measures the strength of an association but does not imply causation.

4.   Samples and Populations

4.1   Understand *sampling* to be the process of selecting a set of individual cases from a larger population to collect data about that population.

4.2   Understand *bias* to be the use of a sampling method that favors some results over others, and describe the different ways in which bias might occur.

4.3   Distinguish between different types of sampling, including convenience sampling, simple random sampling, stratified sampling, and cluster sampling, and describe the merits and drawbacks of each.

4.4   Distinguish between a statistic (data from a sample) and a parameter (data from a population).

Descriptive Statistics

5.   Distribution

5.1   Understand the *distribution* of the data to be the pattern in the way the data are spread across a range of values and how often they take each value.

5.2   Describe the overall pattern of a distribution by its shape, center, and variability.

5.3   Find and interpret different measures of central tendency (mean, median, and mode), and describe the merits and drawbacks of each.

5.4   Find and interpret different measures of variability/dispersion (range, interquartile range, variance, and standard deviation), and describe the merits and drawbacks of each.

5.5   Understand *outliers* as extreme departures from the other values in a distribution.

5.6   Identify outliers and describe the effects outliers have on measures of data (e.g., mean, standard deviation) and the strategies that can mitigate those effects.

5.7   Distinguish between different distribution patterns, including symmetric, skewed, unimodal, bimodal, multimodal, and normal distributions.

5.8   **CHALLENGE TOPIC**: Calculate the arithmetic mean, geometric mean, and harmonic mean, and describe the merits and drawbacks of each.

6. Representation
   6.1   Know the terminology of data representation, including tables, charts, axes, scale, units, data points, increase, decrease, fluctuation, and trend.
   6.2   Represent a data set with a bar graph, a pie chart, a dot plot, a histogram, a frequency table (including relative frequencies), a stem and leaf plot, a box plot, and a scatter plot.
   6.3   Select an appropriate graph or chart to represent a particular set of data.

Probability Theory

7. Probability
   7.1   Understand *probability* as the chance of an individual event happening and as the long-run relative frequency of an event.
   7.2   Know the terminology of probability including event, trial, simulation, chance, probability model, sample space, Venn diagram, intersection, and union.
   7.3   Understand the *Law of Large Numbers* and describe the expected behavior when a chance process is repeated many times.
   7.4   Derive and use basic probability rules including the complement rule, the addition rule for mutually exclusive events, the general addition rule, the general multiplication rule, and the multiplication rule for independent events.
   7.5   Calculate the probabilities of mutually exclusive, overlapping (dependent), and independent combined events.
   7.6   Understand *conditional probability* to be the probability that an event (A) will happen given the knowledge that another event (B) has already happened, and use the conditional probability formula $P\left(B\middle|A\right) = \frac{P(A\,and\,B)}{P(A)}$.
   7.7   Draw and analyze a tree diagram showing the probabilities of a multiple stage process.

8. Combinatorics
   8.1   Derive and use basic counting principles, including the multiplication principle, the addition principle, and the complement principle.
   8.2   Distinguish between permutations and combinations.
   8.3   Derive and use formulas for permutations $(nPr = \frac{n!}{(n-r)})$ and combinations $(nCr = \frac{n!}{r!(n-r)!})$.
   8.4   Count the number of possible arrangements of a group of items using permutations and combinations.

9. Random Variables

    9.1   Understand a *random variable* (X) to be a variable whose numerical value changes depending on the result of a random event.

    9.2   Understand that the *probability distribution* of a random variable gives its possible values and their probabilities.

    9.3   Distinguish between discrete and continuous random variables.

    9.4   Understand the *expected value* of a discrete random variable to be the sum of the value of each outcome of an event weighted by the probability of that outcome (weighted average), and use the expected value formula $EV(x) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + \cdots$.

    9.5   Calculate the standard deviation and variance of a discrete random variable.

    9.6   Describe the effects of various transformations on a random variable, including adding or subtracting a constant, multiplying or dividing by a constant, and adding or subtracting multiple random variables.

    9.7   Distinguish between dependent and independent random variables.

    9.8   Understand a *binomial random variable* to be the count of k successes in n trials of an event given a success probability for each trial, and derive and use the Binomial Probability Theorem: $P(X = k) = nCk * p^k * (1 - p)^{n-k}$.

    9.9   Calculate the mean and standard deviation of a binomial random variable.

    9.10   Understand a *geometric random variable* to be the count of Y trials that it takes to get a success given a success probability p for each trial, and derive and use the Geometric Probability Theorem: $P(Y = k) = (1 - p)^{k-1} * p$.

10. Probability Density Functions

    10.1   Understand a *density curve* to be a means of describing a distribution that has an area of exactly 1 underneath it.

    10.2   Know that the area under part of a density curve (between an upper and lower bound) represents the probability that a randomly selected value from the distribution will fall within that range.

    10.3   Describe the effect of adding, subtracting, multiplying, or dividing by a constant on the shape, center, and variability of a distribution of data.

    10.4   Know that the mean of a density curve is its balance point and that the median of a density curve divides the area in half.

    10.5   Distinguish between different density curves, including normal, uniform, symmetric, and skewed.

11. Distributions

    11.1   Understand the *normal distribution* to be a bell-shaped curve (probability density function) that is symmetric about its mean.

11.2   Describe the kinds of data that might be approximated by a normal distribution (e.g., height, birth weight, job satisfaction).

11.3   Determine whether a distribution of data is approximately normal by visual inspection of a graph or by statistical tests.

11.4   Understand the *standard normal distribution* (z-distribution) to be a normal curve with a mean of 0 and a standard deviation (sigma, σ) of 1, described by the equation $y = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.

11.5   Describe the location of an individual value within a distribution by calculating its percentile and z-score.

11.6   Derive and use the 68-95-99.7 rule (1σ, 2σ, 3σ) to calculate probabilities with normally distributed data.

11.7   Find the area in a normal distribution corresponding to an interval of values for the variable.

11.8   Find the interval of values for the variable in a normal distribution corresponding to a given area (percentile).

11.9   Understand a *t-distribution* to be a symmetric, single-peaked, approximately bell-shaped density curve.

11.10 Understand the *chi-square distributions* to be a series of distributions varying in shape according to the number of independent variables that can be estimated in analysis (degrees of freedom).

Inferential Statistics

12. Estimates and Sample Sizes

12.1   Understand *sampling variability* to refer to the fact that repeated random samples of the same size from the same population produce different statistical measures (e.g., mean, proportion, variance) from each sample.

12.2   Know that sampling variability decreases as sample size increases.

12.3   Understand *sampling distribution* to be the probability distribution of a statistic that comes from drawing random samples of a given population.

12.4   Distinguish between statistics that are biased or unbiased estimators of a parameter.

12.5   Understand the *sample mean* ($\overline{x}$) to be the average of the individual values in a sample.

12.6   Calculate and analyze the mean and standard deviation of the sampling distribution of $\overline{x}$.

12.7   Calculate and analyze the mean and standard deviation of the sampling distribution of the sample proportion ($\hat{p}$).

12.8   Determine when $\hat{p}$ is approximately normal.

12.9   Derive and use the Central Limit Theorem (for large sample sizes, the sampling distribution of the sample means is approximately normal).

13. Confidence Intervals

    13.1   Understand a *confidence interval* to be a range of values that contains the true population parameter a given proportion of the time and understand the *confidence level* to be that given proportion of the time.

    13.2   Know that deciding on a confidence level is a judgement call based on the type of data being analyzed, and know that a commonly used confidence level is 95%.

    13.3   Understand the *margin of error* to be a statistic expressing the amount of random sampling error in an estimate of a population parameter.

    13.4   Understand a *critical value* to be the boundary for claiming significance based on the confidence level.

    13.5   Construct and interpret a confidence interval for a population mean using a z-distribution or t-distribution as appropriate.

    13.6   Construct and interpret a confidence interval for a difference between two population means.

    13.7   Construct and interpret a confidence interval for a population proportion.

    13.8   Construct and interpret a confidence interval for a difference between two population proportions.

    13.9   Construct and interpret a confidence interval for population variance using a chi-square distribution.

    13.10 Calculate the sample size required to obtain a specific confidence level and specific margin of error for a population proportion.

    13.11 Understand *standard error* to be the standard deviation of the sampling distribution.

    13.12 Calculate the standard error of a sample proportion $\hat{p}$ using $SE_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

14. Hypothesis Testing

    14.1   Understand a *hypothesis* to be a testable claim about one or more population parameters.

    14.2   Distinguish between one-sided and two-sided hypotheses.

    14.3   State the null hypothesis (H_0) for a given population parameter, positing that there is no effect or no difference.

    14.4   State an appropriate alternative hypothesis (H_a) for a given population parameter, positing that there is an effect or a difference.

    14.5   Understand a *significance test* to be the statistical method used to decide whether to reject a null hypothesis and by default accept the alternative hypothesis.

    14.6   Know that there are different significance tests, including the z-test, t-test, and chi-square test, which depend on the kinds of data, sample size, and knowledge or lack of knowledge of population variance.

    14.7   Understand the *p-value* of a test (probability value) to be a statistical measure indicating the likelihood that an observed effect or difference is due to chance.

    14.8   Calculate and interpret p-values.

14.9   Distinguish between Type-1 (false-positive) and Type-2 (false-negative) errors.

14.10 Understand the *significance level* (α) to be the threshold against which a p-value is compared in order to decide whether to reject a null hypothesis.

14.11 Know that the confidence level is 1 - α.

14.12 Distinguish between z, t, and chi-squared tests and select an appropriate test for particular types of data.

14.13 Perform tests on population means, including z-tests and t-tests.

14.14 Perform tests on population proportions, including z-tests.

14.15 Perform tests on population variance, including chi-square tests.

14.16 **CHALLENGE TOPIC**: Perform z-tests and t-tests, as appropriate, about claimed differences between two proportions and differences between two means.

14.17 **CHALLENGE TOPIC**: Perform chi-square tests for independence, homogeneity, and goodness of fit on appropriate data.

14.18 **CHALLENGE TOPIC**: Understand the *power* of a test to be the probability that the test will correctly lead to the rejection of a false null hypothesis, and know strategies for increasing the power of a significance test.

Correlation and Regression

15. Association

15.1   Understand two variables to be *associated* if knowing the value of one variable helps us predict the value of the other.

15.2   Describe an association using a scatter plot to show its direction, form, strength, and unusual features.

15.3   Distinguish between two variables that have positive association, negative association, or no association.

15.4   Distinguish between situations where an association implies causation and where an association does not imply causation.

15.5   Understand the *correlation coefficient* (r) as a measure of the strength and direction of a linear association, and describe the merits and drawbacks of correlation coefficients.

15.6   Know that a linear association does not apply to nonlinear relationships.

16. Regression

16.1   Understand a *regression line* to be straight line with the equation $\hat{y} - b_0 + b_1 x$ that describes how a response variable y changes as an explanatory variable x changes.

16.2   Estimate the value of a response variable for a specific value of an explanatory variable using a regression equation.

16.3   Understand *extrapolation* to be a use of a regression equation outside of the interval of values used to obtain the equation, and describe the merits and drawbacks of extrapolation.

16.4   Understand a *residual* as the difference between the actual value of y and the value of $\hat{y}$ predicted by the regression equation.

16.5   Know that a regression line makes the sum of the squared residuals as small as possible.

16.6   Understand a *residual plot* to be a scatter plot displaying the residuals on the vertical axis and the explanatory variable on the horizontal axis.

16.7   Assess how well a regression equation fits a set of data using a residual plot.

16.8   Understand the *coefficient of determination* (r^2) to be a measure of how well a regression equation fits a set of data, specifically the percent of variation in the response variable accounted for by the regression equation using a particular explanatory variable.

16.9   Construct a confidence interval and perform a significance test for the slope of a regression line.

16.10   **CHALLENGE TOPIC**: Linearize a set of non-linear data using mathematical transformations, and determine which transformation produces the most accurate linear relationship.

Mathematical Practices

17.   Students should:

17.1   *Solve word problems using mathematical concepts including*: the use of data in geographical, cultural, scientific, financial, logistical, industrial, meteorological, medical, and natural contexts.

17.2   *Use appropriate physical and conceptual tools including*: the coordinate plane, variables, tables, graphs, charts, diagrams, formulas, statistical models, study design methods, sampling methods, significance tests, and graphing calculators.

17.3   *Understand and use appropriate mathematical language and terminology including*: data, nominal, ordinal, discrete, continuous, increase, decrease, fluctuate, trend, observational study, experiment, factors, significance, testing, power, sampling, confidence, association, correlation coefficient, regression, distribution, and random variables.

17.4   *Create representations of mathematical scenarios, problems, and processes including*: bar graphs, pie charts, dot plots, histograms, frequency tables (including relative frequencies), stem and leaf plots, box plots, scatter plots, distribution graphs, and residual plots.

17.5   *Reason about mathematical relationships and give evidence for conclusions by*: identifying and extracting mathematical information from various sources (e.g., graphs and pictures, verbal descriptions, numerical representations), developing and testing hypotheses about a given set of data, drawing diagrams to illustrate problem contexts, and describing the sensitivity of a given answer to the various factors in the problem.

17.6   *Communicate mathematical thinking by*: designing experiments, collecting and analyzing data, presenting the results in a compelling fashion, relating solutions back to the problem context, and explaining why an answer does or does not make sense.

History of Mathematics

18. Read and discuss true stories of famous mathematicians. Such as:
    18.1   William Playfair, Charles Joseph Minard, Florence Nightingale, and the graphical representation of statistical data.
    18.2   Carl Gauss and the development of the normal curve.
    18.3   Thomas Bayes and Bayes' theorem for conditional probabilities.
    18.4   Karl Pearson, Sir Ronald Fischer, Rudolf Kalman, and the foundation, development, and application of modern statistics.

19. Explore the nature and purpose of mathematics.
    19.1   Describe the ways in which data can be useful in describing reality and the ways in which interpretations of data can be deceptive.
    19.2   Connect mathematics to biology by investigating the way statistics allows researchers to make sense of medical data.
    19.3   Imagine the ways in which the collection of compelling evidence relates to the rhetorical art of persuasion.

Glossary

Absolute Value: The distance of a rational number from zero on a number line.

Addition (+): The arithmetical operation of calculating the total of two or more quantities, often emerging from the processes of putting together, getting more, and counting on.

Angle: A geometrical figure formed by two rays (sides) with the same endpoint (vertex).

Antiderivative: A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Area: The amount of space inside a 2-dimensional shape.

Arithmetic Sequence: A sequence with a constant difference between any two successive terms.

Association: A relationship between two variables where knowing the value of one variable helps predict the value of the other.

Associative Property: A property of the real numbers that states that the order in which numbers are grouped will not change the final sum or product.

Of Addition: $a + (b + c) = (a + b) + c$

Of Multiplication: $a * (b * c) = (a * b) * c$

Axiom (postulate): a statement that is assumed to be true without any kind of proof or justification as the first principles of an axiomatic system. (Example: In the axiomatic system of Geometry, the statement "all right angles are congruent to one another" would be an axiom.)
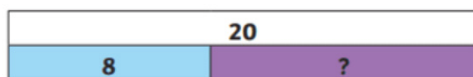
Bar Model (strip diagram): A visual representation of a math problem that uses bars to represent quantities (see picture for example).

Base (exponents): The number that is multiplied by itself the number of times shown by the exponent. (Example: In 5^3, 5 is the base.)

Base (numbers): The number of unique digits, including zero, that a positional numeral system uses to represent numbers. (Example: The decimal system uses base 10, with digits 0-9.)

Bias: The use of a sampling method that favors some results over others.

Binomial Expansions: The result of expanding a binomial expression raised to a power, showing all the terms. (Example: $(x + y)^2$ expands to $x^2 + 2xy + y^2$.)

Binomial Random Variable: The count of k successes in n trials of an event given a success probability p for each trial.

- **Binomial Probability Theorem:** $P(X = k) = nCk * p\,k * (1 - p)\,n{-}k$.

Bisection: A process that divides a figure into two congruent halves.

Capacity: The amount a container can hold, usually measured in units of liquid volume like liters, gallons, or cups.

Cardinality (cardinal numbers): A number that represents the size, or number of elements, within a set (often answering the question "how many?").

Centroid: The intersection point of the three medians of a triangle (sometimes referred to as the center of mass).

Circle: A 2-dimensional shape that is perfectly round made up of the set of all points equidistant from a fixed point called the center.

Closure: A set of numbers is closed under an operation if performing that operation on any number in the set produces another number in the set.

Clustering: Grouping numbers that are close in value together to make it easier to estimate a sum.

Coefficient: A number that is multiplied by a variable.

Combination: The selection of a group of objects from a larger group without regard to order.

Commutative Property: A property of the real numbers that states that the order of the numbers will not change the final sum or product.

- **Of Addition:** $a + b = b + a$
- **Of Multiplication:** $a * b = b * a$

Complex Conjugate: For a complex number of the form $a + bi$, its complex conjugate is $a - bi$.

Complex Number: A number of the form $a + bi$ where a and b are real numbers and i is the imaginary unit.

Composite Number: A number that is divisible by a number other than 1 and the number itself (not prime).

Composition (functions): The application of one function to the results of another function.

Compound (Combined) Inequality: A statement that joins two or more inequalities using the conjunction "and" or "or".

Concept: A unifying mathematical idea or principle that connects different procedures and representations.

Conditional Probability: The probability that an event (B) will happen, given the knowledge that another event (A) has already happened.

Cone: A pyramid with a circular base.

Confidence Interval: A range of values that contains the true population parameter a given proportion of the time.

Congruent: Figures or objects that have the same size and shape.

Conjecture: A statement that is believed to be true based on observation or pattern, but has not yet been proven.

Conjugate Binomials: Two binomials that are identical except for the sign between the terms. (Example: $x + 3$ and $x - 3$ are conjugate binomials.)

Constant: A fixed numerical value that never changes.

Construction: The drawing of a geometric figure using only two instruments, a compass and a straightedge.

Continuity (functions): A function is continuous over an interval when it can be drawn in that interval as a curve with no missing points, breaks, or jumps, that is, when the value of the function at that point is equal to the limit (both left and right hand) of the function as x approaches that point.

Contrapositive: The contrapositive of a conditional statement "If p, then q" is "If not q, then not p." The contrapositive is logically equivalent to the original statement.

Conversion: Changing a measurement from one unit to another equivalent unit (e.g., inches to centimeters, or feet to yards).

Coordinate Plane: A diagram for graphing two related pieces of information with two perpendicular number lines.

Counting Numbers (natural numbers): The set of positive whole numbers starting with 1 and continuing infinitely (1, 2, 3,...).

Cube: A 3-dimensional polyhedron composed of six square faces.

Cube Root: The cube root of a number is the number that produces the original number when multiplied by itself three times.

Cylinder: A 3-dimensional solid that looks like a can with a round tube and a circle at both ends.

Data: Information gathered by counting, measuring, questioning, or observing.

Decimal Number: A number with a whole part and a fractional part separated by a decimal point (.).

Defined Terms: Mathematical objects or relationships that have specific, agreed-upon meanings that can be expressed in terms of undefined or previously defined terms. (Examples: right angle, even number, etc.).

Denominator: The number on the bottom of a fraction representing the number of equal-sized parts the whole is broken into.

Density Curve: A means of describing a distribution that has an area of exactly 1 underneath it.

Diagonal: A segment joining two nonconsecutive vertices of a polygon or polyhedron.

Derivative: The derivative of $f(x)$ is the function $f\prime(x)$ where $f\prime\left(x\right) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = $ if the limit exists. The derivative often represents the slope of a tangent line or the instantaneous rate of change of a function.

Differential Equation: An equation that contains an unknown function and one or more of its derivatives.

Dilation (mathematical): A transformation that changes the size of a figure but not its shape, by multiplying all distances from a center point by a scale factor.

Discriminant: The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$.

Distribution (statistics): The pattern in the way the data are spread across a range of values and how often they take each value.

Distributive Property: A property of the real numbers in which a single value is multiplied across two or more other values within a set of parentheses, namely $a * (b + c) = a * b + a * c$.

Dividend: In a division expression, the number that is being divided. (Example: In 12 ÷ 3 = 4, 12 is the dividend.)

Division (÷): An arithmetical operation that breaks a whole into equal parts or groups.

Divisor: In a division expression, the number that divides the dividend. (Example: In 12 ÷ 3 = 4, 3 is the divisor.)

Domain: The set of inputs accepted by a function.

Edge: A line segment that connects two vertices forming the boundary of a two-dimensional shape (polygon) or three-dimensional shape (polyhedron).

Element: A member of a set.

Equation: A mathematical sentence that relates two equal expressions with an equal sign.

Expanded Form: Writing a number to show the value of each digit. (Example: 325 in expanded form is 300 + 20 + 5.)

Expected Value: The sum of the value of each outcome of an event weighted by the probability of that outcome (weighted average).

- **Expected Value Formula:** $EV(x) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + \cdots$.

Experiment: A study that deliberately imposes some treatment on individuals to measure their responses.

Exponent: Notation describing repeated multiplication of the same number.

Expression: A mathematical phrase that contains numbers, variables, operation signs, and/or grouping symbols such as parentheses but does not contain an equal sign.

Face: A flat surface of a three-dimensional shape.

Factor Theorem: A theorem that states that a polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

Factorial (!): The product of an integer and all the positive integers below it.

Formula: A set of symbols that expresses a mathematical relationship, law, or principle.

Fraction: A part of a whole, expressed as a number divided by another number.

Function: An equation, rule, or law that assigns to each input (independent value) exactly one output (dependent value).

Fundamental Theorem of Algebra: A polynomial of degree n will have exactly n complex roots.

Geometric Sequence: A sequence in which the ratio of any two successive terms is constant.

Golden Ratio: A special irrational number, $(1 + \frac{\sqrt{5}}{2})$, approximately 1.618, often represented by the Greek letter phi (φ), that appears in many geometric and natural contexts.

Greatest Common Factor (GCF): The largest number that divides two or more numbers without leaving a remainder.

Grouping Symbols: Symbols like parentheses ( ), brackets [ ], and braces { } that indicate the order in which operations should be performed.

Hexagon: A 2-dimensional polygon with six sides.

Hypothesis: A testable claim about one or more population parameters.

Imaginary Number: The product of any real number and the imaginary unit i where $i^2 = -1$.

Index: In the context of roots, the index indicates which root is being taken. (Example: In $\sqrt[3]{8}$, 3 is the index, representing the cube root.)

Integers: A number with no decimal or fractional part. The integers are made up of the set of whole numbers and their opposites (..., ...).

Inequality: A mathematical sentence that relates two unequal expressions with the signs >, <, ≥, or ≤.

Integral: A Riemann sum approximating the area under the curve of a function where the number of equal-width rectangles approaches infinity.

Inverse (functions): The inverse of a function is the correspondence from the function's range back to its domain.

Inverse Operations (opposite operations): Operations that "undo" each other, such as addition and subtraction, or multiplication and division.

Irrational Number: A non-terminating, non-repeating decimal that cannot be expressed as a fraction of two integers.

Isometry: A transformation in which every segment in the pre-image is mapped to a congruent segment in the image.

Least Common Multiple (LCM): The smallest number that is a multiple of two or more numbers.

Like Terms: Terms that have the same variable raised to the same power. (Example: $3x$ and $4x$ are like terms.)

Limit: The limit of a function $f(x)$ approaches some value L as x approaches some value a if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to (but not equal to) a.

Line: A one-dimensional object with length but no width or height, extending indefinitely in opposite directions.

Linking Cubes: Small cubes that can be connected together, used to represent numbers and explore mathematical concepts.

Logarithm: The power to which a base must be raised to get another number.

Mapping: A correspondence from one set of points (the pre-image) to another set of points (the image).

Margin of Error: A statistic expressing the amount of random sampling error in an estimate of a population parameter.

Matrix: A rectangular array of numbers enclosed by brackets or parentheses.

Mean: The average of a set of numbers.

- **Arithmetic Mean:** The sum of a set of numbers divided by the count of numbers.
- **Geometric Mean:** For a set of n positive numbers, it is the n^{TH} root of the product of the numbers.
- **Harmonic Mean:** The reciprocal of the average of the reciprocals.

Measurement: The quantification of the characteristics of an object or event by comparing it to a known standard unit.

- **English System:** A system of measurement commonly used in the United States, including units like inches, feet, miles, ounces, pounds, and gallons.
- **Metric System:** A decimal-based system of measurement used globally, including units like meters, grams, and liters, with prefixes like kilo-, centi-, and milli-.

Median (statistics): The middle value in a set of numbers ordered from least to greatest. (If there's an even number of values, the median is the average of the two middle numbers.)

Median (of a triangle): A line segment that connects a vertex of a triangle to the midpoint of the opposite side, dividing that side into two equal parts.

Mode: The value or values that occur most often in a list (or set) of data.

Monomial: A polynomial with only one term.

Multiplication (*): An arithmetical operation that indicates how many times a number is added to itself.

Natural Numbers (counting numbers): The set of positive whole numbers starting with 1 and continuing infinitely (…).

Negative Number: A number less than 0.

Net: A two-dimensional pattern that can be folded to form a three-dimensional shape.

Non-Standard Units: Units of measurement that are not standardized, often based on everyday objects. (Example: measuring length with paper clips or hand spans.)

Normal Distribution: A bell-shaped curve that is symmetric about its mean.

Null: Empty or having no value; often used in the context of the empty set (a set with no elements), denoted by {} or Ø.
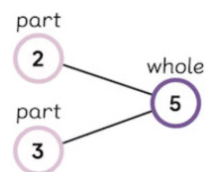
Number Bond: A visual representation showing the relationship between a number and its component parts, usually used to help teach addition and subtraction (see picture for example).

Numeral: A symbol or name that stands for a number. (Example: 5, V, and "five" are all numerals representing the same number.)

Numerator: The number on the top of a fraction representing the number of selected equal-sized parts.

Octagon: A 2-dimensional polygon with eight sides.

Order of Operations: A set of rules that dictate the sequence in which mathematical operations should be performed (Parentheses/Brackets, Exponents/Orders, Multiplication and Division, Addition and Subtraction—PEMDAS/BODMAS).

Ordered Pair: A pair of numbers (x, y) where the order is significant, often used to represent a point on a coordinate plane.

Ordinal Number: A type of number that indicates position or order in a sequence (e.g., first, second, third).

Ordinal Variable: A variable whose possible values have a natural order, such as {short, medium, long}, {cold, warm, hot}, or {0, 1, 2, 3, ...}.

Outlier: A value that is far from most of the others in a data set.

Parallel Lines: Lines in the same plane that never intersect.

Parametric Equation: A relationship where both the dependent variable y and the independent variable x are written as functions of a third variable t (called a parameter).

Parentheses: Grouping symbols ( ) used to enclose expressions and indicate the order of operations.

Partial Sum: The sum of a finite number of terms in a sequence that can often be used to approximate the sum of an infinite series.

Partitive Division: Division where the total number of objects and the number of groups are known, and the goal is to find the number of objects in each group (Sharing).

Part-Part Ratios: A ratio that compares one part of a whole to another part of the same whole. (Example: the ratio of boys to girls in a class.)

Part-Whole Ratios: A ratio that compares a part of a whole to the entire whole. (Example: the ratio of boys to the total number of students in a class.)

Pentagon: A 2-dimensional polygon with five sides.
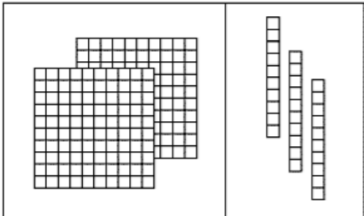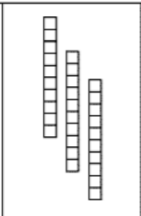
Percentage: A part of a hundred (%).

Perimeter: The distance around a 2-dimensional shape.

Periodic Function: A function that repeats its values at regular intervals or periods, satisfying the equation $f(x + p) = f(x)$ for some constant p.

Permutation: An arrangement of a group of objects in a particular order.

Perpendicular Lines: Lines that intersect at a right angle (90 degrees).

Pi (π): The constant ratio of a circle's circumference to its diameter.

Hundreds	Tens	Ones
2	3	3

Place Value Chart: A table or visual tool that displays the value of each digit in a number based on its position within the number, showing the "place value" of each digit (ones, tens, hundreds, etc.) (see picture for example).

Plane: A two-dimensional object with length and width but no height, extending indefinitely in both dimensions.

Platonic Solids: The five regular, convex polyhedra: tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

Point: A zero-dimensional object with location but no length, width, or height.

Polar Coordinate System: A coordinate system in which points are described by the ordered pair (r, θ) where r is the radial distance from the origin and θ is the angle between the positive x-axis and the line connecting the point to the origin.

Polygon: A closed two-dimensional figure formed by line segments (sides).

Polyhedron: A 3-dimensional solid bounded by faces, edges, and vertices.

Polynomial: A sum of algebraic terms having variables, coefficients, exponents, and/or constants.

Power Series: A series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$.

Prime Number: A number that is only divisible by 1 and itself.

Prime Factorization: Expressing a composite number as a product of its prime factors. (Example: The prime factorization of 12 is $2 \times 2 \times 3$.)

Probability: The likelihood or chance of an event occurring.

Proof: A logical argument that uses established facts (axioms, definitions, previously proven theorems) to demonstrate the truth of a theorem.

Property: A statement that can be proven or shown to be true based on a set of axioms.

Proportion: An equation of two ratios.

Pyramid: A 3-dimensional solid with a flat shape at the base and a single point at the top.

Pythagorean Theorem: In a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides ($a^2 + b^2 = c^2$).

Quadrants (first, second, third, fourth): The four regions of the coordinate plane, numbered counterclockwise starting from the top right.

Quadratic Equation: An equation where the highest exponent of the variable is 2.

Quadrilateral: A 2-dimensional polygon with four sides.

Quotative Division: Division where the total number of objects and the number of objects in each group are known, and the goal is to find the number of groups (Measurement).

Radian: A unit of measure equal to the measure of the central angle of a circle that subtends an arc which is equal in length to the radius of that circle (~57.3˚).

Radical: The symbol $\sqrt{}$ used to indicate a root, most commonly a square root.

Radicand: The number or expression under the radical symbol. (Example: In the expression $\sqrt[4]{81}$, the radicand is 81.)

Random Sample: A subset of a population chosen in such a way that every member of the population has an equal chance of being selected.

Random Variable: A variable whose numerical value changes depending on the result of a random event.

Range (functions): The set of outputs produced by a function.

Range (statistics): The difference between the highest and lowest values in a data set.

Rate: A ratio of two quantities with different units. (Example: 2 eggs for every 3 cups of flour.)

Ratio: A comparison of two quantities.

Rational Numbers: Numbers that can be expressed as a fraction $\frac{p}{q}$, where p and q are integers and q is not zero.

Ray: A part of a line with one endpoint that continues indefinitely in one direction.

Real Number: Any number that can be represented on a number line.

Reciprocal: The reciprocal of a number (other than 0) to be 1 over that number.

Rectangle: A 2-dimensional polygon with four sides and L-shaped corners.

Rectangular Prism: A 3-dimensional polyhedron that looks like a box with six flat faces that are all rectangles.

Reflex Angle: An angle that measures greater than 180 degrees and less than 360 degrees.

Regrouping: The process of rearranging quantities in place value to carry out operations like addition and subtraction (sometimes referred to as "borrowing" or "carrying").

Relation: A collection of ordered pairs determined by an expression, rule, law, or mapping.

Remainder: A leftover of the dividend after the process of division.

Remainder Theorem: If a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

Right Angle: An angle that measures exactly 90 degrees.

Right Triangle: A triangle with one 90° (right) angle.

Rigid Transformation: A change in a shape that generates a congruent shape by preserving distances between vertices.

Root: A value that, when multiplied by itself a certain number of times, equals a given number. (Example: 2 is a square root of 4, and 2 is a cube root of 8.)

Sample Space: The set of all possible outcomes for a situation or experiment.

Sampling: The selection of a smaller group (a set of individual cases) from a larger group (population).

Scaling: Proportionally increasing or decreasing distances in geometrical problems.

Scientific Notation: A way of writing very large or very small numbers as the product of a number between 1 and 10 and a power of 10. (Example: 3,000,000 can be written as $3 * 10^6$.)

Segment: A part of a line between two endpoints.

Semi-circle: Half of a circle.

Sequence: A list of numbers or objects in a particular order.

Series: An expression that adds together the terms of a sequence.

Set: A collection of distinct objects, considered as an object in its own right.

Significance Test: The statistical method used to decide whether to reject a null hypothesis and by default accept the alternative hypothesis.

Significant: The findings of a study are statistically significant when the association between variables cannot be explained by chance.

Similar: Figures that have the same shape but not necessarily the same size; their corresponding angles are equal, and their corresponding sides are in proportion.

Slope: The constant rate of change for a line that is calculated by the vertical change of a line (rise) divided by its horizontal change (run).

Solid: A closed three-dimensional figure.

Sphere: A perfectly round solid shape with no corners or sides.

Square: A rectangle where all the sides are the same length.

Square Root: The square root of a number is the number that when multiplied by itself produces the original number.

Standard Algorithm: A step-by-step procedure for solving addition, subtraction, multiplication, and division problems.

Standard Units: Units of measurement that are formally defined and widely accepted, such as meters, grams, and seconds in the metric system.

Statistic: A number computed from data.

Statistical Measures: Values that summarize aspects of a data set, such as mean, median, mode, range, and standard deviation.

Statistics (field of study): The branch of mathematics dealing with the collection, organization, presentation, analysis, and interpretation of data.

Study: An analysis of data in order to learn about a population.

Subtraction (-): The arithmetical operation of calculating the difference between two quantities, often emerging from the processes of as taking away, taking apart, and counting back.

Surface Area: The combined area of the faces on the outside of a three-dimensional shape.

Symmetry: the property of a shape or object where it appears the same after undergoing a set of transformations (moved, rotated, or flipped).

- **Line (mirror) Symmetry:** A property of a shape or object where one part is a mirror image of another part.
- **Rotational Symmetry :** A property of a shape or object that appears the same after a rotation around a point.
- **Point Symmetry:** A property of a shape or object where it appears the same after being rotated exactly 180 degrees around a central point.
- **Translational Symmetry:** A property of a shape or object where it appears the same after being translated a certain distance in a specific direction.

System of Equations: A set of two or more equations to be solved together containing the same variables.

Tally Chart: A chart that uses tally marks (vertical lines) to record and count frequencies of data.

Theorem (proposition): A mathematical statement that has been proven to be true.

Triangle: A 2-dimensional polygon with three sides.

Undefined Terms: Fundamental terms in an axiomatic system that are not formally defined but are understood intuitively. (Example: in the axiomatic system of Geometry, the terms *point*, *line*, and *plane* are often left undefined.)

Unit: A single, standard quantity used as a basis for measurement.

Unit Circle: The circle with center (0, 0) and radius 1, often used to calculate the value of trigonometric functions.

Unit Fraction: A fraction with a numerator of 1. (Example: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.)

Unit Rate: A ratio with a denominator of 1 (e.g., miles per hour, price per item).

Unlike Terms: Terms that do not have the same variable raised to the same power.

Variable: A letter or other symbol standing for a changeable or unknown quantity.

Vector: A directed line segment with a magnitude and direction that is independent of the coordinate system.

Vertex (geometry): A point where two or more lines, line segments, or edges meet (a corner).

Vertex (graphing): The highest or lowest point on a curve.

Volume: The space inside a three-dimensional object.

Whole Numbers: The set of non-negative integers, starting with zero (0, 1, 2, 3, …).

X-intercept: The point(s) where the graph of an equation crosses the x-axis.

Y-intercept: The point(s) where the graph of an equation crosses the y-axis.